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Volume 3: General Theory of C*-Algebras

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C*-Algebras

VOLUME 3:
GENERAL THEORY OF C*-ALGEBRAS

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C*-Algebras

Volume 3: General Theory of C*-Algebras

Corneliu Constantinescu
Departement Mathematik, ETH Zürich
CH-8092 Zürich
Switzerland

2001
ELSEVIER
Amsterdam - London - New York - Oxford - Paris - Shannon - Tokyo

ELSEVIER SCIENCE B.V.
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P.O. Box 211, 1000 AE Amsterdam, The Netherlands

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First edition 2001

Library of Congress Cataloging in Publication Data
A catalog record from the Library of Congress has been applied for.

ISBN (this volume): 0 444 50751 5
ISBN (5 volume set): 0 444 50758 2
Series ISSN: 0924 6509

♾ The paper used in this publication meets the requirements of ANSI/NISO Z39.48-1992 (Permanence of Paper).
Printed in The Netherlands.

Preface

Functional analysis plays an important role in the program of studies at the Swiss Federal Institute of Technology. At present, courses entitled Functional Analysis I and II are taken during the fifth and sixth semester, respectively. I have taught these courses several times and after a while typewritten lecture notes resulted that were distributed to the students. During the academic year 1987/88, I was fortunate enough to have an eager enthusiastic group of students that I had already encountered previously in other lecture courses. These students wanted to learn more in the area and asked me to design a continuation of the courses. Accordingly, I proceeded during the academic year, following, with a series of special lectures, Functional Analysis III and IV, for which I again distributed typewritten lecture notes. At the end I found that there had accumulated a mass of textual material, and I asked myself if I should not publish it in the form of a book. Unfortunately, I realized that the two special lecture series (they had been given only once) had been badly organized and contained material that should have been included in the first two portions. And so I came to the conclusion that I should write everything anew – and if at all – then preferably in English. Little did I realize what I was letting myself in for! The number of pages grew almost imperceptibly and at the end it had more than doubled. Also, the English language turned out to be a stumbling block for me; I would like to take this opportunity to thank Prof. Imre Bokor and Prof. Edgar Reich for their help in this regard. Above all I must thank Mrs. Barbara Aquilino, who wrote, first a WordMARC™, and then a LaTeX™ version with great competence, angelic patience, and utter devotion, in spite of illness. My thanks also go to the Swiss Federal Institute of Technology that generously provided the infrastructure for this extensive enterprise and to my colleagues who showed their understanding for it.

Corneliu Constantinescu

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Introduction

This book has evolved from the lecture course on Functional Analysis I had given several times at the ETH. The text has a strict logical order, in the style of "Definiton – Theorem – Proof – Example – Exercises". The proofs are rather thorough and there are many examples.

The first part of the book (the first three chapters, resp. the first two volumes) is devoted to the theory of Banach spaces in the most general sense of the term. The purpose of the first chapter (resp. first volume) is to introduce those results on Banach spaces which are used later or which are closely connected with the book. It therefore only contains a small part of the theory, and several results are stated (and proved) in a diluted form. The second chapter (which together with Chapter 3 makes the second volume) deals with Banach algebras (and involutive Banach algebras), which constitute the main topic of the first part of the book. The third chapter deals with compact operators on Banach spaces and linear (ordinary and partial) differential equations – applications of the theory of Banach algebras.

The second part of the book (the last four chapters, resp. the last three volumes) is devoted to the theory of Hilbert spaces, once again in the general sense of the term. It begins with a chapter (Chapter 4, resp. Volume 3) on the theory of C^*–algebras and W^*–algebras which are essentially the focus of the book. Chapter 5 (resp. Volume 4) treats Hilbert spaces for which we had no need earlier. It contains the representation theorems, i.e. the theorems on isometries between abstract C^*–algebras and the concrete C^*–algebras of operators on Hilbert spaces. Chapter 6 (which together with Chapter 7 makes Volume 5) presents the theory of $\mathcal{L}^p$–spaces of operators, its application to the self–adjoint linear (ordinary and partial) differential equations, and the von Neumann algebras. Finally, Chapter 7 presents examples of C^*–algebras defined with the aid of groups, in particular the Clifford algebras. Many important domains of C^*–algebras are ignored in the present book. It should be emphasized that the whole theory is constructed in parallel for the real and for the complex numbers, i.e. the C^*–algebras are real or complex.

In addition to the above (vertical) structure of the book, there is also a second (horizontal) division. It consists of a main strand, eight branches, and additional material. The results belonging to the main strand are marked with (0). Logically speaking, a reader could restrict himself/herself to these and ignore the rest. Results on the eight subsidiary branches are marked with (1), (2), (3), (4), (5), (6), (7), and (8). The key is

1. Infinite Matrices
2. Banach Categories
3. Nuclear Maps
4. Locally Compact Groups
5. Differential Equations
6. Laurent Series
7. Clifford Algebras
8. Hilbert C^*–Modules

These are (logically) independent of each other, but all depend on the main strand. Finally, the results which belong to the additional material have no marking and – from a logical perspective – may be ignored. So the reader can shorten for himself/herself this very long book using the above marks. Also, since the proofs are given with almost all references, it is possible to get into the book at any level and not to read it linearly.

We assume that the reader is familiar with classical analysis and has rudimentary knowledge of set theory, linear algebra, point–set topology, and integration theory. The book addresses itself mainly to mathematicians, or to physicists interested in C^*–algebras.

I would like to apologize for any omissions in citations occasioned by the fact that my acquaintance with the history of functional analysis is, unfortunately, very restricted. For this history we recommend the following texts.

1. BIRKHOFF, G. and KREYSZIG, E., The Establishment of Functional Analysis, Historia Mathematica 11 (1984), 258–321.

2. BOURBAKI, N., Elements of the History of Mathematics, (21. Topological Vector Spaces), Springer–Verlag (1994).

3. DIEUDONNÉ, J., History of Functional Analysis, North–Holland (1981).

4. DIEUDONNÉ, J., A Panorama of Pure Mathematics (Chapter C III: Spectral Theory of Operators), Academic Press (1982).

5. HEUSER, H., Funktionalanalysis, 2. Auflage (Kapitel XIX: Ein Blick auf die werdende Functionalanalysis), Teubner (1986), 3. Auflage (1992).

6. KADISON, R.V., Operator Algebras, the First Forty Years, in: Proceedings of Symposia in Pure Mathematics 38 I (1982), 1–18.

7. MONNA, A.F., Functional Analysis in Historical Perspective, John Whiley & Sons (1973).

8. STEEN, L.A., Highlights in the History of Spectral Theory, Amer. Math. Monthly 80 (1973), 359–382.

There is no shortage of excellent books on C^*–algebras. Nevertheless, we hope that this book will be also of some utility to the mathematics community.

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VOLUME 3:
GENERAL THEORY OF C*-ALGEBRAS

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4. C^*–Algebras

In this chapter we leave the realm of operators on Banach spaces and hence that of Banach algebras for the realm of operators on Hilbert spaces. The C^*–algebras play the same role with respect to these operators as did the Banach algebras with respect to the operators on Banach spaces. We therefore formulate abstractly the algebra of operators on Hilbert spaces. But in contrast to Banach algebras, every C^*–algebra is isometric to a closed involutive subalgebra of the algebra of operators on a Hilbert space.

The principal difficulty constantly confronted in the study of C^*–algebras is the lack of commutativity. It would not be unfair to say that to certain extent all that has been done in the theory of C^*–algebras consists of exploiting a small commutative toe–hold, namely selfadjointness and normality, in this non–commutative world to find locally commutative C^*–algebras which are embedded in the hostile non–commutative C^*–algebras. Once this toe–hold has been consolidated, the commutative theory can be applied in the non–commutative case as well, for example by using the absolute value.

4.1 The General Theory

Three central results on C^*–algebras are proved in this section. First we show that these algebras are symmetric, which implies that the Gelfand transform is an isometry. Next we use this result to develop a functional calculus for self–normal operators, which is the principal tool in the theory. Finally the Fuglede–Putnam Theorem enables us to enlarge the toe–hold of commutativity provided by self–normal operators to commutative sets of self–normal operators.

4.1.1 General Results

Definition 4.1.1.1 (0) (Gelfand–Naimark, 1943). *A **complex (unital) C^*-algebra** is an involutive (unital) complex Banach algebra E such that*

$$\|x^*x\| = \|x\|^2$$

*for every $x \in E$. A **real (unital) C^*-algebra** is an involutive (unital) real Banach algebra E such that its complexification $\overset{\circ}{E}$ (Proposition 2.3.1.40) admits a norm extending the norm of E (identified with $E \times \{0\}$) and making $\overset{\circ}{E}$ a complex C^*-algebra (by Corollary 4.1.1.21 this norm on $\overset{\circ}{E}$ is unique). A C^*-**algebra** is a complex or a real C^*-algebra.*

*Let E be a (unital) C^*-algebra. A **(unital) C^*-subalgebra** of E is a closed involutive (unital) subalgebra of E. It is a (unital) C^*-algebra with respect to the structure induced from E (Proposition 2.3.1.40). The smallest (unital) C^*-subalgebra of E containing a given subset A of E is called **the (unital) C^*-subalgebra of E generated by A**. A **Gelfand C^*-algebra** is a C^*-algebra which is an involutive Gelfand algebra.*

Every closed involutive ideal of a C^*-algebra E is a C^*-subalgebra of E.

If E is a C^*-algebra then

$$\|x\|^2 \leq \|x^*x + y^*y\| \qquad (*)$$

for all $x, y \in E$ (Corollary 4.2.1.18, Theorem 4.2.2.1 c $\Rightarrow$ a). Conversely, it can be shown, that if E is an involutive real algebra such that $(*)$ holds, then E is a real C^*-algebra (T.W. Palmer, Pacific Journal of Math. 35 (1970) 195-204).

If E is a unital C^*-algebra, then $1 + x^*x$ is invertible for every $x \in E$ (Theorem 4.2.2.1 c $\Rightarrow$ a). Conversely, it can be shown, that if E is an involutive unital real Banach algebra such that

$$\|x^*x\| = \|x\|^2$$

and $1 + x^*x$ is invertible for every $x \in E$, then E is a real C^*-algebra (K.R. Goodearl, Notes on Real and Complex C^*-Algebras, Shiva Publishing Limited, 1982).

If E is a real C^*-algebra, then the identity map $E \times E \to \overset{\circ}{E}$ is an isomorphism of Banach spaces. Indeed

$$\|x + iy\| \leq \|x\| + \|y\|$$

for every $(x, y) \in E \times E$, so that the above map is continuous. By the Principle of Inverse Operator, this map is an isomorphism.

The real Banach algebra $\mathbb{C}$ endowed with the involution

$$\mathbb{C} \longrightarrow \mathbb{C}, \quad x \longmapsto x$$

is not a real C^*–algebra although

$$\|x^*x\|) \|x\|^2$$

holds for all $x \in \mathbb{C}$.

Remark. If E is a C^*–algebra and $x, y \in E$ then:

a) $x^*xy = 0 \Rightarrow xy = 0$;

b) $yxx^* = 0 \Rightarrow yx = 0$;

c) $x^*xy = x^*x \Rightarrow xy = x$;

d) $yxx^* = xx^* \Rightarrow yx = x$.

a) Since $x^*xy = 0$, $y^*x^*xy = 0$, i.e.

$$\|xy\|^2 = \|(xy)^*(xy)\| = \|y^*x^*xy\| = 0.$$

Hence $xy = 0$.

b) Since $yxx^* = 0$, $xx^*y^* = 0$. Hence $x^*y^* = 0$ by a). It follows $yx = 0$.

c) $\|xy - x\|^2 = \|(xy - x)^*(xy - x)\| = \|(y^*x^* - x^*)(xy - x)\| =$

$$= \|y^*x^*xy - x^*xy - y^*x^*x + x^*x\| = \|y^*(x^*xy - x^*x) - (x^*xy - x^*x)\| = 0,$$

$xy = x$.

d) We have $xx^*y^* = xx^*$. By c), $x^*y^* = x^*$. Hence $yx = x$.

Proposition 4.1.1.2 (0) *Let E be an involutive algebra and f a positive real function on E such that for every $x, y \in E$,*

$$f(xy) \leq f(x)f(y), \qquad f(x)^2 \leq f(x^*x).$$

a) *For all $x \in E$,*

$$f(x) = f(x^*), \qquad f(x)^2 = f(x^*x) = f(xx^*).$$

b) *For all* $x \in \mathrm{No}\, E$ *and* $n \in \mathbb{N}$,

$$f(x^n) = f(x)^n .$$

c) *If* E *possesses a unit then*

$$f(1) \in \{0, 1\} .$$

a) We have successively that

$$f(x)^2 \leq f(x^*x) \leq f(x^*)f(x) ,$$

$$f(x) \leq f(x^*) ,$$

$$f(x^*) \leq f(x^{**}) = f(x) ,$$

$$f(x) = f(x^*) ,$$

$$f(x^*x) \leq f(x^*)f(x) = f(x)^2 ,$$

$$f(x)^2 = f(x^*x) ,$$

$$f(xx^*) = f(x^{**}x^*) = f(x^*)^2 = f(x)^2 .$$

b) By a)

$$f(x)^4 = f(x^*x)^2 = f(x^*xx^*x) = f((x^*)^2x^2) = f((x^2)^*x^2) = f(x^2)^2,$$

such that

$$f(x^2) = f(x^2) .$$

Let $n \in \mathbb{N}$ and assume

$$f(x^{2^n}) = f(x)^{2^n} .$$

Since x^{2^n} is normal, we get

$$f(x^{2^{n+1}}) = f((x^{2^n})^2) = f(x^{2^n})^2 = f(x)^{2^{n+1}}$$

by the above considerations. Hence, by complete induction,

$$f(x^{2^n}) = f(x)^{2^n}$$

for all $n \in \mathbb{N}$. By Proposition 2.2.3.1.

$$\inf_{n\in\mathbb{N}} f(x^n)^{\frac{1}{n}} = \lim_{n\to\infty} f(x^n)^{\frac{1}{n}} = \lim_{n\to\infty} f(x^{2^n})^{2^{-n}} = f(x)\,,$$

$$f(x) \leq f(x^n)^{\frac{1}{n}} \leq f(x)\,,$$

$$f(x^n) = f(x)^n\,.$$

c) If E possesses a unit, then by a),

$$f(1)^2 = f(1^*1) = f(1)$$

so that $f(1) \in \{0,1\}$. ■

Proposition 4.1.1.3 (0) *Let E be an involutive (unital) algebra which is a Banach algebra with respect to the norm $\|\cdot\|$. Suppose further that*

$$\|x\|^2 \leq \|x^*x\|$$

for every $x \in E$. Then E is a (unital) involutive Banach algebra in which

$$\|x\|^2 = \|x^*x\| = \|xx^*\|$$

for every $x \in E$.

The assertion follows from Proposition 4.1.1.2 a). ■

Remark. a) The relation

$$\|x^*x\| = \|xx^*\|$$

is an important drop of commutativity in the uncommutative world of involutive Banach algebras.

b) Let E be as in the proposition. The proposition shows that the relation

$$x \in E \Longrightarrow \|x^*x\| = \|x\|^2$$

implies the relation

$$x \in E \Longrightarrow \|x^*x\| = \|x^*\|\,\|x\|\,.$$

The reverse implication holds too for $\mathbb{K} = \mathbb{C}$, but the proof is long.

Proposition 4.1.1.4 (0) *The unitary elements of a unital C^*-algebra have norm 1.*

Let x be a unitary element of a unital C^*-algebra. Then

$$\|x\|^2 = \|x^*x\| = \|1\| = 1\,.$$ ■

Example 4.1.1.5 (0) *If T is a set then $\ell^\infty(T)$ is a unital C^*-algebra, $c(T)$ is a unital C^*-subalgebra of $\ell^\infty(T)$, and $c_0(T)$ is a C^*-subalgebra of $c(T)$. If T is a topological space, then $\mathcal{C}(T)$ is a unital C^*-subalgebra of $\ell^\infty(T)$; if T is locally compact then $\mathcal{C}_o(T)$ is a C^*-subalgebra of $\mathcal{C}(T)$. If $\mathfrak{R}$ is a σ-algebra on T, then*

$$\{x \in \ell^\infty(T) \mid x \text{ is } \mathfrak{R}\text{-measurable}\}$$

is a unital C^*-subalgebra of $\ell^\infty(T)$. If $(T, \mathfrak{T}, \mu)$ is a measure space, then the involutive commutative unital Banach algebra $L^\infty(\mu)$ (Example 2.3.2.3) is a C^*-algebra. ■

Remark. If T is an infinite set and $p \in [1, \infty[$, then there is no norm on the involutive subalgebra $\ell^p(T)$ of $\ell^\infty(T)$ which renders it a C^*-algebra. Indeed, assume we had such a norm. Let $(A_n)_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{P}_f(T)$ such that

$$\operatorname{Card} A_n = 2^n$$

for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$,

$$\|e_{A_n}\|^2 = \|e_{A_n}^* e_{A_n}\| = \|e_{A_n}\|\,,$$

so that

$$\|e_{A_n}\| = 1\,.$$

Thus the sequence $\left(\frac{1}{n^2} e_{A_n}\right)_{n\in\mathbb{N}}$ is absolutely summable. Let x be its sum. Then

$$x(t) = \frac{1}{n^2}$$

for every $n \in \mathbb{N}$ and $t \in A_n$ and we obtain the contradiction

$$\infty = \sum_{n\in\mathbb{N}} \frac{2^n}{n^{2p}} \leq \sum_{t\in T} |x(t)|^p < \infty\,.$$

Example 4.1.1.6 (0) *Let $(E_t)_{t\in T}$ be a family of (unital) C^*-algebras and let*

$$E := \{x \in \prod_{t\in T} E_t \mid (\|x_t\|)_{t\in T} \in \ell^\infty(T)\}$$

be the involutive normed (unital) algebra defined in Example 2.3.2.4). Then E is a (unital) C^*-algebra (called ***the C^*-direct product*** *of the family $(E_t)_{t\in T}$) and*

$$\{x \in E \mid (\|x_t\|)_{t\in T} \in c_0(T)\}$$

is an involutive closed ideal of E (and therefore a C^*-algebra), called ***the C^*-direct sum*** *of the family $(E_t)_{t\in T}$. If T is a topological space and all E_t $(t \in T)$ are equal, then*

$$\{x \in E \mid x \text{ is continuous}\}$$

is a (unital) C^*-subalgebra of E. ■

Example 4.1.1.7 (0) *For every $n \in \mathbb{N}$ the involutive unital Banach algebra $\mathcal{L}(\mathbb{K}^n)$ (Example 2.3.2.6) is a unital C^*-algebra. For every set T, the involutive unital C^*-algebra $\mathcal{L}(\ell^2(T))$ (Example 2.3.2.27 c)) is a unital C^*-algebra. For every measure space $(T, \mathfrak{T}, \mu)$, the involutive unital Banach algebra $\mathcal{L}(L^2(\mu))$ (Example 2.3.2.27 c)) is a unital C^*-algebra.*

The last two assertions follow from Example 2.3.2.27 c), d). The first assertion is a special case of the second one. ■

Remark. In general, neither of the involutive Banach algebras $\ell^{2,2}(T,T)$ (Example 2.3.2.5) and $L^2(\mu \otimes \mu)$ (Proposition 3.1.6.18 b)) is a C^*-algebra.

Theorem 4.1.1.8 (0) *Let E be a complex (unital) C^*-algebra and $\overset{\circ}{E}$ (resp. $\overbrace{E'}^{\circ}$) the complexification of the underlying involutive real algebra of E (resp. involutive vector space of E'). We denote by $\overset{\leftrightarrow}{E}$ the complex (unital) C^*- algebra obtained from E by reversing the multiplication, i.e. by endowing E with the multiplication*

$$E \times E \longrightarrow E, \quad (x,y) \longmapsto yx.$$

a) $\overset{\circ}{E}$ *becomes a complex (unital)* C^**-algebra with respect to the norm*

$$\overset{\circ}{E} \longrightarrow \mathbb{R}_+ , \quad (x,y) \longmapsto \sup\{\|x+iy\| , \|x-iy\|\} .$$

In particular if we restrict the scalars of E *to* $\mathbb{R}$ *then* E *is a real (unital)* C^**-algebra. A real* C^**-algebra which is not of this form (i.e. which is not obtained from a complex* C^**-algebra by the restrictions of the scalars to* $\mathbb{R}$*) will be called* ***purely real*** C^****-algebra***.

b) *The map*

$$\overset{\circ}{E} \longrightarrow E \times \overset{\leftrightarrow}{E} , \quad (x,y) \longmapsto ((x+iy),(x-iy)^*)$$

is an isometry of complex (unital) C^**-algebras (a) and Example 4.1.1.6).*

c) $\overset{\circ}{\widehat{E'}}$ *is an involutive complex Banach space with respect to the norm*

$$\overset{\circ}{\widehat{E'}} \longrightarrow \mathbb{R}_+ , \quad (x',y') \longmapsto \|x'+iy'\| + \|x'-iy'\| .$$

d) *Given* $(x',y') \in \overset{\circ}{\widehat{E'}}$ *, define*

$$\widetilde{(x',y')} : \overset{\circ}{E} \longrightarrow \mathbb{C} , \quad (x,y) \longmapsto \langle x+iy, x'+iy' \rangle + \overline{\langle x-iy, x'-iy' \rangle} .$$

Then for each $(x',y') \in \overset{\circ}{\widehat{E'}}$, $\widetilde{(x',y')} \in (\overset{\circ}{E})'$ *and*

$$\widetilde{(x',y')} \in (\overset{\circ}{E})'_+ \Longleftrightarrow x'+iy' , \; x'-iy' \in E'_+ ,$$

and the map

$$\overset{\circ}{\widehat{E'}} \longrightarrow (\overset{\circ}{E})' , \quad (x',y') \longmapsto \widetilde{(x',y')}$$

is an isometry of involutive Banach spaces.

e) *If* $u : E \to E$ *is an isomorphism of real* C^**-algebras which is conjugate linear, then the map*

$$\overset{\circ}{E} \longrightarrow E \times E , \quad (x,y) \longmapsto (x+iy, ux+iuy)$$

is an isomorphism of complex C^**-algebras.*

f) *If T is a topological space then the map*

$$\overset{\circ}{\widehat{\mathcal{C}(T)}} \longrightarrow \mathcal{C}(T) \times \mathcal{C}(T)\,, \quad (x,y) \longmapsto (x+iy, \overline{x}+i\overline{y})$$

is an isomorphism of complex C^*-algebras. If T is locally compact, then we may replace $\mathcal{C}$ by $\mathcal{C}_0$ in the above map.

a),b),c),d) By Example 2.3.2.35 and 2.3.4.22, we need only prove that $\overset{\circ}{E}$ is a complex C^*-algebra. Take $(x,y) \in \overset{\circ}{E}$. We have

$$\|(x,y)^*(x,y)\| = \|(x^*,-y^*)(x,y)\| = \|(x^*x+y^*y\,,\, x^*y-y^*x)\| =$$

$$= \sup\{\|(x^*x+y^*y)+i(x^*y-y^*x)\|, \|(x^*x+y^*y-i(x^*y-y^*x)\|\} =$$

$$= \sup\{\|(x+iy)^*(x+iy)\|\,,\, \|(x-iy)^*(x-iy)\|\} =$$

$$= \sup\{\|x+iy\|^2\,,\, \|x-iy\|^2\} = \|(x,y)\|^2\,.$$

e) Put

$$v : E \longrightarrow \overset{\leftrightarrow}{E}\,, \quad x \longmapsto u^{-1}x^*\,.$$

Since u^{-1} is an isomorphism of real C^*-algebras which is conjugate linear, v is an isomorphism of complex C^*-algebras (Proposition 2.3.1.50). Then v^{-1} is an isomorphism of complex C^*-algebras and

$$v(uy)^* = u^{-1}uy = y\,,$$

$$v^{-1}y = (uy)^*$$

for every $y \in \overset{\leftrightarrow}{E}$. Hence for $x,y \in E$,

$$v^{-1}(x-iy)^* = (u((x-iy)^*)^* = u(x-iy) = ux+iuy\,.$$

By b), then map

$$\overset{\circ}{E} \longrightarrow E \times E\,, \quad (x,y) \longmapsto (x+iy, ux+iuy)$$

is an isomorphism of complex (unital) algebras.

f) follows from e). ■

Remark. A unital real C^*-algebra E is not purely real iff there is an $x \in E^c$ with

$$x^* = -x\,, \quad x^2 = -1\,.$$

Example 4.1.1.9 *Let E be a complex C^*-algebra, A an algebraic basis of the underlying complex vector space of E, and F the real vector subspace of E generated by A. If F is an involutive real subalgebra of E, then the map*

$$\overset{\circ}{F} \longrightarrow E\,, \quad (x,y) \longmapsto x + iy$$

is a isomorphism of involutive complex algebras. If, in addition, F is closed, then F is a real C^*-algebra and E may be identified with its complexification. ■

Example 4.1.1.10 *Let E be a commutative complex C^*-algebra and endow $\operatorname{Re} E$ with the induced structure of an involutive real algebra. Then the map*

$$u : \overset{\circ}{\widehat{\operatorname{Re} E}} \longrightarrow E\,, \quad (x,y) \longmapsto x + iy$$

is an isomorphism of involutive complex algebras. In particular, $\operatorname{Re} E$ is a real C^*-algebra and E may be identified with its complexification. ■

Example 4.1.1.11 *Let E be a complex unital C^*-algebra. Take $x \in E$ such that*

$$x^2 = -1\,, \quad x^* = -x\,,$$

and that x and 1 are linearly independent. Put

$$F := \{\alpha 1 + \beta x \mid \alpha, \beta \in \mathbb{R}\}\,.$$

Then F is an involutive commutative real subalgebra of E and a division algebra and the map

$$\overset{\circ}{F} \longrightarrow E(x)\,, \quad (y,z) \longmapsto y + iz$$

is an isomorphism of involutive complex algebras. In particular, F is a commutative real C^*-algebra, which is not a Gelfand algebra, and $E(x)$ may be identified with the complexification of F. The map

$$\overset{\circ}{\mathbb{C}} \longrightarrow E(x)\,, \quad (\alpha,\beta) \longmapsto \alpha 1 + \beta x$$

is an isometry of C^*-algebras (Example 4.1.1.8).

$\{1, x\}$ is an algebraic basis of the underlying complex vector space of $E(x)$. By Example 4.1.1.9, F is a real C^*-algebra and $E(x)$ may be identified with the complexification of F. Since

$$i \in \sigma_{\overset{\circ}{F}}((x,0))\,,$$

F is not a Gelfand algebra. ■

Remark. $x := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{C}^2)$ possesses the above properties.

Proposition 4.1.1.12 (0) *If E is a C^*-algebra, then*

$$\|x\| = \sup_{y \in E^\#} \|xy\|$$

for every $x \in E$.

We may assume that $\|x\| = 1$. Then $\|x^*\| = 1$, so that

$$\sup_{y \in E^\#} \|xy\| \geq \|xx^*\| = \|x\|^2 = 1\,.$$

The reverse inequality is trivial. ■

Theorem 4.1.1.13 (0) (B. Yood) *Let E be a C^*-algebra and $\mathbb{K} \times E$ the associated involutive unital algebra of E. We identify E with the involutive ideal $\{0\} \times E$ of $\mathbb{K} \times E$. Then $\mathbb{K} \times E$ endowed with the norm*

$$\mathbb{K} \times E \longrightarrow \mathbb{R}_+\,, \quad x \longrightarrow \sup_{y \in E^\#} \|xy\| = \sup_{y \in E^\#} \|yx\|$$

if E has no unit and with the norm

$$\mathbb{K} \times E \longrightarrow \mathbb{R}_+\,, \quad (\alpha, x) \longmapsto \sup\{|\alpha|\,, \|x + \alpha 1_E\|\}$$

if E has a unit 1_E is a unital C^*-algebra inducing on E its given norm. If E has no unit, then the above $\mathbb{K} \times E$ is called ***the unital C^*-algebra associated to*** *E. If E has a unit, then* ***the unital C^*-algebra associated to*** *E will be E itself.*

Case 1 E has no unit

The map

$$\mathbb{K} \times E \longrightarrow \mathbb{R}_+\,, \quad x \longmapsto \sup_{y \in E^\#} \|xy\|$$

is obviously a seminorm. By Proposition 4.1.1.12, it induces on E the original norm of E. Assume that

$$\sup_{y \in E^\#} \|(\alpha, x)y\| = 0$$

for some element $(\alpha, x) \in \mathbb{K} \times E$, $(\alpha, x) \neq (0,0)$. Then $\alpha \neq 0$ and

$$\sup_{y \in E^\#} \|\alpha y + xy\| = 0 .$$

Hence

$$\alpha y + xy = 0 ,$$

$$-\frac{1}{\alpha} xy = y$$

for every $y \in E$. Hence $-\frac{1}{\alpha} x$ is a unit of E (Proposition 2.3.1.13 c)), contradicting our hypothesis. Thus the map

$$\mathbb{K} \times E \longrightarrow \mathbb{R}_+ , \quad x \longmapsto \sup_{y \in E^\#} \|xy\|$$

is a norm. Since E is complete, this norm is also complete (Corollary 1.2.5.6).

Let $x, y \in \mathbb{K} \times E$ and $z \in E$. Then $yz \in E$ and

$$\|yz\| \leq \|y\| \, \|z\| .$$

Hence

$$\|(xy)z\| = \|x(yz)\| \leq \|x\| \, \|yz\| \leq \|x\| \, \|y\| \, \|z\|$$

and

$$\|xy\| \leq \|x\| \, \|y\| .$$

Hence $\mathbb{K} \times E$ endowed with the above norm is a unital Banach algebra .

Take $x \in \mathbb{K} \times E$. We want to prove that

$$\|x\|^2 \leq \|x^* x\| .$$

We may assume that $\|x\| = 1$. Take $\alpha < 1$. There is a $y \in E^\#$ with

$$\|xy\|^2 > \alpha .$$

Since $xy, x^* xy \in E$, it follows that

$$\alpha < \|xy\|^2 = \|(xy)^*(xy)\| = \|y^* x^* xy\| \leq \|y^*\| \, \|x^* xy\| \leq \|x^* x\| .$$

Since α is arbitrary, we deduce that

$$\|x\|^2 = 1 \leq \|x^*x\| .$$

By Proposition 4.1.1.3, $\mathbb{K} \times E$ is a C^*–algebra with respect to the above norm.

Take $x \in \mathbb{K} \times E$. Then

$$\|x\| = \|x^*\| = \sup_{y \in E^\#} \|x^*y\| = \sup_{y \in E^\#} \|(y^*x)^*\| = \sup_{y \in E^\#} \|y^*x\| = \sup_{y \in E^\#} \|yx\| .$$

Case 2 E has a unit

Let F be the C^*–algebra $\mathbb{K} \times E$ defined in Example 4.1.1.6. Put

$$u : \mathbb{K} \times E \longrightarrow F , \quad (\alpha, x) \longmapsto (\alpha, \alpha 1 + x) .$$

Then u is linear and

$$u((\alpha, x)(\beta, y)) = u(\alpha\beta, \alpha y + \beta x + xy) = (\alpha\beta, \alpha\beta 1 + \alpha y + \beta x + xy) =$$

$$= (\alpha\beta, (\alpha 1 + x)(\beta 1 + y)) = (\alpha, \alpha 1 + x)(\beta, \beta 1 + y) = u(\alpha, x)u(\beta, y) ,$$

$$u(\alpha, x)^* = (\alpha, \alpha 1 + x)^* = (\overline{\alpha}, \overline{\alpha} 1 + x^*) = u(\overline{\alpha}, x^*) = (u(\overline{\alpha}, x)^*)$$

for every (α, x), $(\beta, y) \in \mathbb{K} \times E$. Hence u is a bijective involutive algebra homomorphism. The map defined in the enunciation of the theorem is the transported norm of F, and $\mathbb{K} \times E$ normed with it is a C^*–algebra inducing on E its initial norm. ■

Remark. Let E be a complex C^*–algebra and let F be the underlying real C^*–algebra of E (Theorem 4.1.1.8 a)). If E is not unital and if $\widetilde{E}$, $\widetilde{F}$ denote the unital C^*–algebras associated to E and F, respectively, then $\widetilde{F}$ is purely real and so it is different from the underlying real C^*–algebra of $\widetilde{E}$.

Convention *The above result allows us to use the unit for C^*–algebras even if the C^*–algebra is not unital. In this case, it is understood that the unit belongs to the associated unital C^*–algebra.*

Corollary 4.1.1.14 *Let T be a locally compact non– compact space and T^* its Alexandroff compactification. Extend each function in $\mathcal{C}_0(T)$ continuously to T^*. Then $\mathcal{C}(T^*)$ is the unital C^*–algebra associated to $\mathcal{C}_0(T)$.* ■

Remark. Put

$$R := [-1,+1]\,, \quad S :=]-1,+1]\,, \quad T := [-1,0[\,\cup\,]0,+1]\,.$$

Then R is homeomorphic to the Alexandroff compactifications of S and T, respectively. By the above corollary, $\mathcal{C}(R)$ is isometric to the unital C^*-algebras associated to $\mathcal{C}_0(S)$ and $\mathcal{C}_0(T)$, respectively. Since $\mathcal{C}_0(S)$ and $\mathcal{C}_0(T)$ are not isometric, it follows that non-isometric, non-unital C^*-algebras may have isometric associated unital C^*-algebras.

Proposition 4.1.1.15 (0) *If E is a complex C^*-algebra then*

$$\sigma(x) \subset \mathbb{R}$$

for every $x \in \operatorname{Re} E$.

We may assume E to be unital (Theorem 4.1.1.13).

Step 1 $\quad i \notin \sigma(x)$

Assume that $i \in \sigma(x)$. Take $\alpha \in \mathbb{R}_+$ and put

$$P := \alpha - it \in \mathbb{C}[t]\,.$$

Then

$$\alpha + 1 = \alpha - i^2 = P(i) \in P(\sigma(x)) = \sigma(P(x)) = \sigma(\alpha 1 - ix)$$

(Theorem 2.1.3.4 b)) so that

$$\alpha + 1 \leq r(\alpha 1 - ix) \leq \|\alpha 1 - ix\|$$

(Proposition 2.2.4.1). We deduce that

$$\alpha^2 + 2\alpha + 1 = (\alpha+1)^2 \leq \|\alpha 1 - ix\|^2 = \|(\alpha 1 - ix)^*(\alpha 1 - ix)\| =$$

$$= \|(\alpha 1 + ix)(\alpha 1 - ix)\| = \|\alpha^2 1 - i^2 x^2\| \leq \|\alpha^2 1\| + \|i^2 x^2\| = \alpha^2 + \|x^2\|\,,$$

$$2\alpha + 1 \leq \|x^2\|\,,$$

which is a contradiction.

Step 2 $\quad \sigma(x) \subset \mathbb{R}$

Take $\alpha, \beta \in \mathbb{R}$ with $\alpha + i\beta \in \sigma(x)$. Assume that $\beta \neq 0$. Put

$$P := \frac{1}{\beta}(t - \alpha) \in \mathbb{C}[t].$$

Then

$$i = \frac{1}{\beta}(\alpha + i\beta - \alpha) = P(\alpha + i\beta) \in P(\sigma(x)) = \sigma(P(x)) = \sigma(\frac{1}{\beta}(x - \alpha 1))$$

(Theorem 2.1.3.4 b)) which contradicts Step 1. Hence $\beta = 0$ and $\sigma(x) \subset \mathbb{R}$. ■

Theorem 4.1.1.16 (0) *Let x be a self-normal element of a C^*-algebra.*

a) $\|x^n\| = \|x\|^n$ *for every* $n \in \mathbb{N}$.

b) $r(x) = \|x\|$.

c) (x *quasinilpotent*) $\Leftrightarrow (\sigma(x) = \{0\}) \Leftrightarrow (x = 0)$.

a) follows from Proposition 4.1.1.2 b).

b) If the C^*-algebra E is real then

$$\sigma_{\overset{\circ}{E}}((x, 0)) \subset \mathbb{R}$$

(Proposition 4.1.1.15). By a) and Theorem 2.2.5.4,

$$r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} = \|x\|.$$

c) By b) and Proposition 2.2.4.21 d),

$$(x = 0) \Longrightarrow (x \text{ quasinilpotent}) \Longrightarrow (\sigma(x) = \{0\}) \Longrightarrow$$

$$\Longrightarrow (r(x) = 0) \Longrightarrow (\|x\| = 0) \Longrightarrow (x = 0).$$ ■

Remark. The condition that x be self-normal cannot be dropped. Indeed, $\mathcal{L}(\mathbb{K}^2)$ is a C^*-algebra (Example 4.1.1.7) and

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathbb{K}^2).$$

We have $x^2 = 0$ so that $\sigma(x) = \{0\}$ (Proposition 2.1.3.16 c)).

Corollary 4.1.1.17 *Let E be a C^*-algebra and take $x, y \in E$. If xy is self-normal, then*

$$\|xy\| \leq \|yx\|.$$

We may assume E to be unital (Theorem 4.1.1.13). Then

$$\|xy\| = r(xy) = r(yx) \leq \|xy\|$$

(Theorem 4.1.1.16 b), Proposition 2.1.3.10 c), Proposition 2.2.4.1). ■

Corollary 4.1.1.18 *Every C^*-algebra is semi-simple.*

Let E be a C^*-algebra with radical R. Take $x \in R$. Then $x^*x \in R$ and by Corollary 2.1.3.24,

$$\sigma(x^*x) = \{0\}\,.$$

By Theorem 4.1.1.16 c),

$$\|x\|^2 = \|x^*x\| = 0\,,$$

so that $x = 0$ and hence $R = \{0\}$. ■

Corollary 4.1.1.19 *Let E be a unital C^*-algebra, x a self-normal element of E, $\alpha \in \mathbb{K}$, and $r > 0$. If*

$$\sigma(x) \subset U_r^{\mathbb{K}}(\alpha)\,,$$

then

$$\|x - \alpha 1\| < r\,.$$

We have that

$$\sigma(x - \alpha 1) = \sigma(x) - \alpha \subset U_r^{\mathbb{K}}(0)\,,$$

and so

$$\|x - \alpha 1\| = r(x - \alpha 1) < r$$

by Theorem 4.1.1.16 b). ■

Corollary 4.1.1.20 (0) (Rickart, 1946) *Let E be an involutive Banach algebra, F a C^*-algebra, and $u : E \to F$ an involutive algebra homomorphism. Then u is continuous and $\|u\| \leq 1$.*

Given $x \in \operatorname{Re} E$,

$$\|ux\| = r_F(ux) \leq r_E(x) \leq \|x\|$$

(Proposition 2.3.1.7, Theorem 4.1.1.16 b), Corollary 2.1.3.12, and Propositions 2.2.4.1, 2.2.1.3).

Take $x \in E$. Then x^*x is selfadjoint, and so

$$\|ux\|^2 = \|(ux)^*(ux)\| = \|u(x^*x)\| \leq \|x^*x\| \leq \|x^*\|\,\|x\| = \|x\|^2$$

by the above remark. Hence

$$\|ux\| \leq \|x\|\,. \qquad \blacksquare$$

Remark. On the basis of this corollary, we find it convenient to call **homomorphism of C^*–algebras** every involutive algebra homomorphism of two C^*–algebras.

Corollary 4.1.1.21 (0) *Let E, F be C^*–algebras and $u : E \to F$ a bijective involutive algebra homomorphism. Then u is an isometry. In particular, the isomorphisms and the isometries of C^*–algebras coincide and there is at most one norm on an involutive algebra making it to a C^*–algebra.* $\blacksquare$

Remark. Let E be a C^*–algebra and $\overset{\leftrightarrow}{E}$ the C^*–algebra obtained from E by reversing the multiplication. By the above corollary and by Proposition 2.3.1.50, E and $\overset{\leftrightarrow}{E}$ are isomorphic iff there is an isomorphism of real C^*–algebras $E \to E$ which is conjugate linear. In particular if $\mathbb{K} = \mathbb{R}$ then E and $\overset{\leftrightarrow}{E}$ are isomorphic.

Proposition 4.1.1.22 *Let E be an involutive normed complex algebra and $\mathcal{P}$ the set of seminorms on E, for which*

$$p(xy) \leq p(x)p(y)\,,$$

$$p(x)^2 \leq p(x^*x)\,,$$

$$p(x) \leq \|x\|$$

for every $x, y \in E$. Define

$$q : E \longrightarrow \mathbb{R}_+\,, \quad x \longmapsto \sup_{p \in \mathcal{P}} p(x)\,.$$

a) $q \in \mathcal{P}$.

b) $\overset{-1}{q}(0)$ *is an involutive ideal of* E.

c) *We have that*

$$q(x) = q(y)$$

for every $A \in E/\overset{-1}{q}(0)$ *and* $x, y \in A$. *Define*

$$\rho(A) := q(x).$$

d) *The map*

$$E/\overset{-1}{q}(0) \longrightarrow \mathbb{R}_+, \quad A \longmapsto \rho(A)$$

defines a norm on $E/\overset{-1}{q}(0)$.

e) $E/\overset{-1}{q}(0)$ *is an involutive normed complex algebra. The quotient map* $E \to E/\overset{-1}{q}(0)$ *is continuous with norm not exceeding* 1 *and the completion of* $E/\overset{-1}{q}(0)$ *is a complex* C^**-algebra, called* ***the*** C^****-hull of*** E.

f) *Every involutive algebra homomorphism of* E *into a complex* C^**-algebra may be factored through the* C^**-hull of* E. *This factorization is an involutive algebra homomorphism and it is unique in the class of involutive algebra homomorphism.*

a) is easy to prove.
b) follows from a) and Proposition 4.1.1.2 a).
c) and d) are easy to see.
e) Given $A, B \in E/\overset{-1}{q}(0)$,

$$\rho(AB) \leq \rho(A)\rho(B),$$

$$\rho(A)^2 \leq \rho(A^*A).$$

By b), Proposition 2.3.1.21 c) and Proposition 4.1.1.2 a), $E/\overset{-1}{q}(0)$ is an involutive normed algebra. The above inequalities hold also for the completion of $E/\overset{-1}{q}(0)$. By Proposition 4.1.1.3, this completion is a C^*-algebra.

f) Let $u : E \to F$ be an involutive algebra homomorphism of E into a C^*-algebra F. Define

$$p : E \longrightarrow \mathbb{R}_+ , \quad x \longmapsto \|ux\| .$$

By Corollary 4.1.1.20, $p \in \mathcal{P}$. Hence

$$\overset{-1}{q}(0) \subset \operatorname{Ker} u .$$

Thus u may be factored through $E/\overset{-1}{q}(0)$. This factorization is obviously an involutive algebra homomorphism. Since it is continuous (Corollary 4.1.1.20), it may be extended continuously to the completion of $E/\overset{-1}{q}(0)$. By continuity, this extention is an involutive algebra homomorphism. The uniqueness follows from Corollary 4.1.1.20. ■

Example 4.1.1.23 *Let E be a finite-dimensional symmetric, involutive, semi-simple, complex Gelfand algebra and u the Gelfand transform on E. Then the C^*-hull of E is E endowed with the norm*

$$E \longrightarrow \mathbb{R}_+ , \quad x \longmapsto \|ux\| .$$

By Proposition 2.4.2.3 a $\Rightarrow$ c, u is involutive and by Corollary 2.4.1.14, u is injective. Since E is finite-dimensional, u is bijective and the assertion now follows from Corollary 4.1.1.21. ■

Proposition 4.1.1.24 (0) *Let E be a unital C^*-algebra, U the topological group of unitary elements of E (Proposition 2.3.1.8, Theorem 2.2.4.14), and $\mathcal{F}$ the topological group of unital isometries $E \to E$ of C^*-algebras (Example 2.2.2.4). Given $x \in E$, define*

$$\widetilde{x} : E \longrightarrow E , \quad y \longmapsto xyx^* .$$

Then $\widetilde{x} = \widetilde{x}^* \in \mathcal{F}$ for every $x \in U$, the map

$$U \longrightarrow \mathcal{F} , \quad x \longmapsto \widetilde{x}$$

is a continuous group homomorphism, and

$$E^c = \{x \in E \mid \widetilde{x} = \textit{identity map}\} = \{x \in U \mid \widetilde{x} = \textit{identity map}\} .$$

Take $x \in U$. Then $\widetilde{x}$ is linear, bijective,

$$\widetilde{x}1 = x1x^* = 1 ,$$

$$\widetilde{x}y^* = xy^*x^* = (xyx^*)^* = (\widetilde{x}y)^* ,$$

$$\widetilde{x}^* y = (\widetilde{x} y^*)^* = \widetilde{x} y$$

for every $y \in E$, and

$$\widetilde{x}(yz) = xyzx^* = xyx^* xzx^* = (\widetilde{x}y)(\widetilde{x}z),$$

for all $y, z \in E$. Hence $\widetilde{x}$ is an involutive unital algebra homomorphism and $\widetilde{x} = \widetilde{x}^*$. By Corollary 4.1.1.21, $\widetilde{x}$ is an isometry of C^*–algebras.

Take $x, y \in U$. Then

$$\widetilde{xy}z = xyzy^* x^* = \widetilde{x}\widetilde{y}z$$

for every $z \in E$. Hence

$$\widetilde{xy} = \widetilde{x}\widetilde{y}$$

and the map

$$U \longrightarrow \mathcal{L}(E), \quad x \longmapsto \widetilde{x}$$

is a group homomorphism. Given $z \in E$,

$$\|(\widetilde{x} - \widetilde{y})(z)\| = \|xzx^* - yzy^*\| \leq \|xzx^* - yzx^*\| + \|yzx^* - yzy^*\| \leq$$

$$\leq \|x - y\| \, \|z\| \, \|x^*\| + \|y\| \, \|z\| \, \|x^* - y^*\| = 2\|x - y\| \, \|z\|$$

(Proposition 4.1.1.4), so that

$$\|\widetilde{x} - \widetilde{y}\| \leq 2\|x - y\|$$

and the map

$$U \longrightarrow \mathcal{F}, \quad x \longmapsto \widetilde{x}$$

is continuous.

We now prove the last assertion. If $x \in E^c$, then

$$\widetilde{x}y = xyx^* = yxx^* = y$$

for every $y \in E$. Hence $\widetilde{x}$ is the identity map. Take $x \in E$ with $\widetilde{x}$ the identity map. Then for any $y \in E$

$$y = \widetilde{x}y = xyx^*.$$

In particular, $xx^* = 1$ and

$$x^*x = x(x^*x)x^* = (xx^*)(xx^*) = 1\,,$$

i.e. $x \in U$. It follows

$$yx = xy$$

for every $y \in E$, i.e. $x \in E^c$. ■

Proposition 4.1.1.25 (0) *Let E be a C^*-algebra, $(F_\iota)_{\iota\in I}$ a family of C^*-subalgebras of E such that*

$$F_\iota F_\lambda = \{0\}$$

for all distinct $\iota, \lambda \in I$, F the C^*-subalgebra of E generated by $\bigcup_{\iota\in I} F_\iota$, and $\sum_{\iota\in I} F_\iota$ the C^*-direct sum of the family $(F_\iota)_{\iota\in I}$. Then $(x_\iota)_{\iota\in I}$ is summable for every $x \in \sum_{\iota\in I} F_\iota$ and the map

$$\sum_{\iota\in I} F_\iota \longrightarrow F\,, \quad x \longmapsto \sum_{\iota\in I} x_\iota$$

is an isometry of C^*-algebras.

If I is finite, then the assertion is obvious since the map

$$u : \sum_{\iota\in I} F_\iota \longrightarrow F\,, \quad x \longmapsto \sum_{\iota\in I} x_\iota$$

is a bijective involutive algebra homomorphism and therefore an isometry of C^*-algebras (Corollary 4.1.1.21). In particular

$$\left\| \sum_{\iota\in I} x_\iota \right\| = \sup_{\iota\in I} \|x_\iota\|$$

for every $(x_\iota)_{\iota\in I} \in \prod_{\iota\in I} F_\iota$.

Now suppose I is infinite. By the above considerations, $(x_\iota)_{\iota\in I}$ is summable for every $x \in \sum_{\iota\in I} F_\iota$ and

$$\|x\| = \|\sum_{\iota\in I} x_\iota\|\,.$$

The map

$$u : \sum_{\iota\in I} F_\iota \longrightarrow F\,, \quad x \longmapsto \sum_{\iota\in I} x_\iota$$

is an involutive norm-preserving algebra homomorphism, so that $\operatorname{Im} u$ is a C^*-subalgebra of F. Since it contains $\bigcup_{\iota\in I} F_\iota$ it is, in fact, F and u is an isometry of C^*-algebras. ■

Proposition 4.1.1.26 (0) *If E is a C^*-algebra such that $\operatorname{Re} E = \{0\}$, then $E = \{0\}$.*

Step 1 $x \in E \Rightarrow x^* = -x$

We have

$$x^* + x \in \operatorname{Re} E = \{0\},$$

so that

$$x^* = -x.$$

Step 2 $x \in E \Rightarrow x^2 = 0$

By the first step,

$$x^2 = -(x^2)^* = -(x^*)^2 = -x^2$$

so that

$$x^2 = 0.$$

Step 3 $x \in E \Rightarrow x = 0$

By the first two steps,

$$\|x\|^2 = \|x^* x\| = \| - x^2\| = 0,$$

so that

$$x = 0.$$ ■

Proposition 4.1.1.27 (0) *Let E be a real C^*-algebra. Take $x' \in (\overset{\circ}{E})'$, and consider*

$$\varphi : E \longrightarrow \overset{\circ}{E}, \quad x \longmapsto (x, 0).$$

a) $(x, y) \in \overset{\circ}{E} \Rightarrow \|(x, y)\| \leq \|x\| + \|y\|$.

b) $x' \in \operatorname{Re}(\overset{\circ}{E})' \Leftrightarrow \varphi' x' \in \operatorname{Re} E'$.

c) $x' \in (\overset{\circ}{E})'_+ \Leftrightarrow \varphi' x' \in E'_+$.

d) *If* $\mathfrak{F}$ *is an approximate unit of* E *then* $\varphi(\mathfrak{F})$ *is an approximate unit of* $\overset{\circ}{E}$ *and*

$$x' \in (\overset{\circ}{E})'_+ \Longrightarrow \|x'\| = \|\varphi' x'\|\,,$$

$$x' \in \tau(\overset{\circ}{E}) \Longleftrightarrow \varphi' x' \in \tau(E)\,.$$

a) We have

$$\|(x,y)\| = \|\varphi x + i\varphi y\| \leq \|\varphi x\| + \|\varphi y\| = \|x\| + \|y\|\,.$$

b) φ is involutive and so φ' is also involutive (Proposition 2.3.2.22 d)) and we get that

$$x' \in \operatorname{Re}(\overset{\circ}{E})' \Longrightarrow \varphi' x' \in \operatorname{Re} E'\,.$$

Assume $\varphi' x' \in \operatorname{Re} E'$ and take $(x,y) \in \operatorname{Re} \overset{\circ}{E}$. Then

$$x^* = x\,, \quad y^* = -y\,,$$

so that

$$\langle (x,y), x' \rangle = \langle \varphi x + i\varphi y, x' \rangle = \langle \varphi x, x' \rangle + i\langle \varphi y, x' \rangle = \langle x, \varphi' x' \rangle + i\langle y, \varphi' x' \rangle \in \mathbb{R}$$

and $x' \in \operatorname{Re}(\overset{\circ}{E})'$ (Corollary 2.3.1.23 b $\Rightarrow$ a).

c) Take $x' \in (\overset{\circ}{E})'_+$. By b), $\varphi' x' \in \operatorname{Re} E'$. For every $x \in E$,

$$\langle x^* x, \varphi' x' \rangle = \langle \varphi(x^* x), x' \rangle = \langle (\varphi x)^* (\varphi x), x' \rangle \in \mathbb{R}_+\,,$$

and so $\varphi' x' \in E'_+$.

Assume $\varphi' x' \in E'_+$. By b), $x' \in \operatorname{Re}(\overset{\circ}{E})'$. For every $(x,y) \in \overset{\circ}{E}$,

$$\langle (x,y)^*(x,y), x' \rangle = \langle (x^* x + y^* y, x^* y - y^* x), x' \rangle =$$

$$= \langle \varphi(x^* x + y^* y), x' \rangle + i\langle \varphi(x^* y - y^* x), x' \rangle =$$

$$= \langle x^* x + y^* y, \varphi' x' \rangle + i\langle x^* y - y^* x, \varphi' x' \rangle \in \mathbb{R}_+\,,$$

so that $x' \in (\overset{\circ}{E})'_+$.

d) Take $(x,y) \in \overset{\circ}{E}$ and $z \in E$. By a),

$$\|(x,y) - (x,y)\varphi z\| = \|(x,y) - (xz, yz)\| = \|(x - xz, y - yz)\| \leq \|x - xz\| + \|y - yz\|\,.$$

It follows that

$$\lim_{z,\mathfrak{F}} \|(x,y) - (x,y)\varphi z\| = 0\,,$$

i.e.

$$\lim_{z,\mathfrak{F}} (x,y)\varphi z = (x,y)\,.$$

Similarly,

$$\lim_{z,\mathfrak{F}} (\varphi z)(x,y) = (x,y)\,.$$

Hence $\varphi(\mathfrak{F})$ is an approximate unit of $\overset{\circ}{E}$.

Take $x' \in (\overset{\circ}{E})'_+$. By Proposition 2.3.4.10 a),

$$\|x'\| = \lim_{x,\mathfrak{F}} \langle (\varphi x)^*(\varphi x), x'\rangle = \lim_{x,\mathfrak{F}} \langle \varphi(x^*x), x'\rangle = \lim_{x,\mathfrak{F}} \langle x^*x, \varphi' x'\rangle = \|\varphi' x'\|\,.$$

By c),

$$x' \in \tau(\overset{\circ}{E}) \Longleftrightarrow \varphi' x' \in \tau(E)\,. \qquad \blacksquare$$

Example 4.1.1.28 *Put*

$$u := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\,, \qquad v := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\,.$$

Then

$$\sup\{\|u\|, \|v\|\} < \|u^*u + v^*v\|^{\frac{1}{2}} < \|(u,v)\| = \|u\| + \|v\|$$

(Examples 4.1.1.7).

We have (Example 1.2.2.8)

$$\|u\|^2 = \frac{1}{2}\left(1+1+1+1+\sqrt{(1+1+1+1)^2 - 4|1+1|^2}\right) = 2\,,$$

$$\|v\|^2 = \frac{1}{2}\left(1+1+1+1+\sqrt{(1+1+1+1)^2 - 4|1+1|^2}\right) = 2\,,$$

$$\|(u,v)\|^2 =$$

$$= \frac{1}{2}\left(2+2+2+2+\sqrt{(2+2+2+2)^2 - 4|(1+i)^2 + (1-i)^2|^2}\right) = 8\,,$$

$$\|u^*u + v^*v\| = 4\,. \qquad \blacksquare$$

Remark. This example shows that the inequality of Proposition 4.1.1.27 a) is sharp.

Proposition 4.1.1.29 *Let $(E_\iota)_{\iota\in I}$ be a family of Gelfand C^*-algebras and E its C^*-direct sum. Given $\iota \in I$ and $x'_\iota \in \sigma(E_\iota)$, define*

$$\widetilde{x'_\iota} : E \longrightarrow \mathbb{K}, \quad x \longmapsto \langle x_\iota, x'_\iota \rangle .$$

Let $\sum_{\iota\in I} \sigma(E_\iota)$ denote the the topological sum of the family of topological spaces $(\sigma(E_\iota))_{\iota\in I}$.

a) *E is a Gelfand C^*-algebra.*

b) *$\widetilde{x'_\iota} \in \sigma(E)$ for every $\iota \in I$ and $x'_\iota \in \sigma(E_\iota)$; we set*

$$\varphi_\iota : \sigma(E_\iota) \longrightarrow \sigma(E), \quad x'_\iota \longmapsto \widetilde{x'_\iota}$$

for every $\iota \in I$.

c) *The map*

$$\varphi : \sum_{\iota\in I} \sigma(E_\iota) \longrightarrow \sigma(E)$$

defined by

$$\varphi|\sigma(E_\iota) = \varphi_\iota$$

for every $\iota \in I$ is a homeomorphism.

a) follows from Proposition 4.1.1.15.

b) is easy to see.

c) It is clear that φ is injective and continuous. For $\iota \in I$ and $x_\iota \in E_\iota$ let $\widetilde{x}_\iota$ be the element of E defined by

$$\widetilde{x}_\iota : I \longrightarrow \bigcup_{\lambda\in I} E_\lambda, \quad \lambda \longmapsto \begin{cases} x_\iota & \text{if } \iota = \lambda \\ 0 & \text{if } \iota \neq \lambda . \end{cases}$$

Take $x' \in \sigma(E)$. Given $\iota \in I$, define

$$x'_\iota : E_\iota \longrightarrow \mathbb{K}, \quad x_\iota \longmapsto x'(\widetilde{x}_\iota) .$$

There is a $\iota \in I$ such that $x'_\iota \neq 0$. Then $x'_\iota \in \sigma(E_\iota)$ and there is an $x_\iota \in E_\iota$ with

$$x'_\iota(x_\iota) \neq 0 .$$

Let $\lambda \in I\backslash\{0\}$. Then for every $x_\lambda \in E_\lambda$, we have

$$\widetilde{x}_\iota \widetilde{x}_\lambda = 0\,,$$

so that

$$0 = x'(\widetilde{x}_\iota \widetilde{x}_\lambda) = x'(\widetilde{x}_\iota)x'(\widetilde{x}_\lambda) = x'_\iota(x_\iota)x'(\widetilde{x}_\lambda)\,,$$

$$x'(\widetilde{x}_\lambda) = 0\,.$$

It follows that

$$x' = \widetilde{x}'_\iota$$

and the map φ is surjective. The above construction shows that φ^{-1} is also continuous. ■

Example 4.1.1.30 *Assume* $\mathbb{K} = \mathbb{C}$. *Let* T *be the Cantor set and*

$$T_1 := T \cap [0, \frac{1}{3}]\,, \quad T_2 := T \cap [\frac{2}{3}, 1]\,.$$

Given $x \in \mathcal{C}(T)$, *define*

$$x_1 : T_1 \longrightarrow \mathbb{C}\,, \quad t \longmapsto x(3t)\,,$$

$$x_2 : T_2 \longrightarrow \mathbb{C}\,, \quad t \longmapsto x(3t-2)\,.$$

Given $(x, y) \in \overset{\circ}{\widetilde{\mathcal{C}(T)}}$, *define*

$$\widetilde{(x,y)} : T \longrightarrow \mathbb{C}\,, \quad t \longmapsto \begin{cases} x_1(t) + iy_1(t) & \text{if } t \in T_1 \\ x_2(t) - iy_2(t) & \text{if } t \in T_2\,. \end{cases}$$

Then the map

$$\overset{\circ}{\widetilde{\mathcal{C}(T)}} \longrightarrow \mathcal{C}(T)\,, \quad (x, y) \longmapsto \widetilde{(x,y)}$$

is an isomorphism of complex C^**-algebras (Theorem 4.1.1.8 a)).* ■

Example 4.1.1.31 (0) *The involutive real algebra* $\mathbb{H}$ *of quaternions endowed with the Euclidean norm*

$$\mathbb{H} \longrightarrow \mathbb{R}_+\,, \quad \alpha + \beta i + \gamma j + \delta k \longmapsto (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^{\frac{1}{2}}$$

is a simple real C^**-algebra.*

The assertion follows from Example 2.3.1.46 (and Example 4.1.1.7, Corollary 2.1.4.17). ■

Example 4.1.1.32 *Let E be a C^*-algebra. If we replace the multiplication in E by the multiplication*

$$x * y := -xy$$

for all $x, y \in E$, then we get a C^*-algebra.

The assertion follows from Example 2.4.3.8 a),g). ■

Proposition 4.1.1.33 (0) *Let $(x_\iota)_{\iota \in I}$ be a family in a C^*-algebra E. Then the following are equivalent:*

a) *$(x_\iota^* x_\iota)_{\iota \in I}$ is absolutely summable.*

b) *$(x_\iota x_\iota^*)_{\iota \in I}$ is absolutely summable.*

If E is finite-dimensional, then the above assertions are equivalent to the following ones:

c) *$(x_\iota^* x_\iota)_{\iota \in I}$ is summable.*

d) *$(x_\iota x_\iota^*)_{\iota \in I}$ is summable.*

a $\Leftrightarrow$ b follows from

$$\iota \in I \Longrightarrow \|x_\iota^* x_\iota\| = \|x_\iota x_\iota^*\| .$$

a $\Leftrightarrow$ c and b $\Leftrightarrow$ d follows from Proposition 1.1.6.14. ■

Remark. If E is infinite-dimensional, then c) and d) may not be equivalent (see Remark of Proposition 5.3.2.13).

4.1.2 The Symmetry of C^*–Algebra

Theorem 4.1.2.1 (0) (Kaplanski) *Every C^*–algebra is strongly symmetric.*

It is sufficient to prove the theorem for $\mathbb{K} = \mathbb{C}$ (Proposition 2.3.1.40). Let E be a C^*–algebra. We may assume E unital (Theorem 4.1.1.13). By Proposition 4.1.1.15,

$$\sigma(x) \subset \mathbb{R}$$

for every $x \in \operatorname{Re} E$. By Corollary 2.4.2.4 b $\Rightarrow$ a, E is symmetric. By Theorem 4.1.1.16 b),

$$r(x) = \|x\|$$

for every $x \in \operatorname{Re} E$, so that the assertion follows from Proposition 2.4.2.9 $c_3 \Rightarrow c_1$. ■

Corollary 4.1.2.2 (0) *A commutative real C^*–algebra all of whose elements are selfadjoint is a Gelfand C^*–algebra. In particular, a C^*–subalgebra of a Gelfand C^*–algebra is a Gelfand C^*–algebra.*

Let E be a commutative real C^*–algebra such that $\operatorname{Re} E = E$. By Theorem 4.1.2.1, $\overset{\circ}{E}$ is symmetric. Thus by Proposition 2.3.1.29,

$$\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R}$$

for every $x \in E$. Hence E is a Gelfand C^*–subalgebra of E. ■

Corollary 4.1.2.3 (0) *Let E be a C^*–algebra and A a commutative subset of E. If $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$), then we assume that $A \cup A^*$ is commutative (that $A \subset \operatorname{Re} E$). Then the C^*–subalgebra of E generated by A is a Gelfand C^*–algebra. Moreover, A is contained in a maximal Gelfand C^*–subalgebra of E.*

The complex case follows from Corollary 2.3.2.14 a),b), so we may assume that $\mathbb{K} = \mathbb{R}$. It is clear that the subalgebra of E generated by A is commutative and that all of its elements are selfadjoint. Its closure, F, has the same properties. By Corollary 4.1.2.2, F is thus a Gelfand C^*–algebra.

The last assertion follows from Zorn's Lemma (and Corollary 2.2.1.10 and Proposition 2.3.2.11). ■

Corollary 4.1.2.4 (0) *Let E be a (unital) C^*-algebra and take x a self-normal element of E with $E = E(x)$ $(E = E(x,1))$. Then the map*

$$\sigma_0(E)\ (\sigma(E)) \longrightarrow \sigma(x)\,, \quad x' \longmapsto x'(x)$$

is a homeomorphism.

By Theorem 4.1.2.1 and Corollary 4.1.2.3, $E(x,1)$ is a symmetric involutive (unital) Gelfand algebra, so the assertion follows from Proposition 2.4.2.10. ■

Corollary 4.1.2.5 (0) (Gelfand, Naimark, 1943) *The Gelfand transform of any (unital) Gelfand C^*-algebra is an isometry of (unital) C^*-algebras. In particular, a Gelfand C^*-algebra E is unital iff $\sigma(E)$ is compact.*

Let E be a Gelfand C^*-algebra. By Theorem 4.1.2.1, E is symmetric and by Theorem 4.1.1.16 b),

$$r(x) = \|x\|$$

for every $x \in E$. By Corollary 2.4.2.7 a $\Rightarrow$ b and Theorem 2.4.1.13, the Gelfand transform of E is an isometry of C^*-algebras. ■

Remark. M.H. Stone proved (1940) the above result for Gelfand C^*-subalgebras of the C^*-algebra of operators on a given Hilbert space.

Corollary 4.1.2.6 *Let E be a (unital) C^*-algebra and F a (unital) Gelfand C^*-subalgebra of E. Then the following are equivalent:*

a) *F is separable.*

b) *There is an $x \in \operatorname{Sn} E$ such that $F = E(x)$ $(F = E(x,1))$.*

If F is finite-dimensional then

$$\operatorname{Dim} F = \operatorname{Card} \sigma(x)\,.$$

In particular, if $E = \mathbb{K}_{n,n}$ for some $n \in \mathbb{N}$, then

$$\operatorname{Dim} F \leq n\,.$$

a $\Rightarrow$ b. We may assume E and F unital. By Theorem 2.4.1.3 d), $\sigma(F)$ is metrizable. Hence there is a $y \in \mathcal{C}(\sigma(F))$ generating $\mathcal{C}(\sigma(F))$ as C^*-algebra (Theorem of Weierstrass-Stone). By Corollary 4.1.2.5, there is an $x \in F$ such that $F = E(x,1)$.

b $\Rightarrow$ a follows from Corollary 2.3.2.14 c).

The final assertion follows from Corollary 4.1.2.4. ■

Corollary 4.1.2.7 (0) *Let E be a complex (real) C^*-algebra. Take $x \in E$ and $x' \in E'$ ($x' \in \operatorname{Re} E'$).*

a) *If E is unital, then $\sigma(x) \subset \overset{\vee}{x}(\tau(E))$.*

b) *If E is unital and x self-normal then $\overset{\vee}{x}(\tau(E))$ is the convex hull of $\sigma(x)$, the extreme points of $\sigma(x)$ are contained in $\overset{\vee}{x}(\tau_0(E))$, there is an $x' \in \tau_0(E)$ such that*

$$|x'(x)| = \|x\|,$$

and

$$\|x\| = \|\overset{\vee}{x}\| = \|\overset{\vee}{x} \mid \tau_0(E)\|.$$

c) *If E is unital then*

$$x' \in E'_+ \Longleftrightarrow x'(1) = \|x'\|.$$

d) *If $x' \in \operatorname{Re} E$, then there are $y', z' \in E'_+$ such that*

$$x' = y' - z', \quad \|x'\| = \|y'\| + \|z'\|.$$

e) $E = \{0\} \Leftrightarrow \tau(E) = \emptyset$.

a & b & c & d. By Theorem 4.1.2.1, E is strongly symmetric and by Theorem 4.1.1.16 b)

$$r(y) = \|y\|$$

for every self-normal element y of E. With the exception of a) for $\mathbb{K} = \mathbb{R}$, all other assertions follow from Proposition 2.3.5.15 a $\Rightarrow$ b & c & d & e, Corollary 2.3.4.7 and Proposition 2.3.5.10 c) (and Theorem 4.1.1.13).

We now prove a) for $\mathbb{K} = \mathbb{R}$. Take $\alpha \in \sigma_E(x)$. Consider

$$\varphi : E \longrightarrow \overset{\circ}{E}, \quad x \longmapsto (x, 0).$$

By Corollary 2.1.5.14, $\alpha \in \sigma_{\overset{\circ}{E}}(\varphi x)$. By the above proof, there is an $y' \in \tau(\overset{\circ}{E})$ such that

$$\langle \varphi x, y' \rangle = \alpha.$$

By Proposition 4.1.1.27 d), $\varphi' y' \in \tau(E)$, so

$$\alpha = \langle \varphi x, y' \rangle = \langle x, \varphi' y' \rangle \in \overset{\vee}{x}(\tau(E)) .$$

e) Assume that $E \neq \{0\}$. By Proposition 4.1.1.26, $\operatorname{Re} E \neq \{0\}$. Take $x \in \operatorname{Re} E \backslash \{0\}$. By Proposition 2.3.2.22 l), there is an $x' \in \operatorname{Re} E'$ such that

$$\|x'\| = 1 , \quad x'(x) = \|x\| .$$

By d), $\tau(E) \neq \{0\}$.

The reverse implication is trivial. ■

Remark. a) holds even for $x' \in E'$ (Proposition 4.3.6.1.g)).

Corollary 4.1.2.8 *Let F be a C^*-subalgebra of the unital complex (real) C^*-algebra E. Take $x' \in E'$ ($x' \in \operatorname{Re} E'$) such that*

$$y' := x'|F \in F'_+ .$$

a) *If $1 \in F$ then*

$$x' \in E'_+ \Longrightarrow \|x'\| = \|y'\| .$$

b) *If F is unital then*

$$\|x'\| = \|y'\| \Longrightarrow x' \in E'_+ .$$

a) By Corollary 4.1.2.7 c),

$$y'(1) = x'(1) = \|x'\| \geq \|y'\| \geq y'(1) ,$$

so that

$$\|x'\| = \|y'\| .$$

b) Denote by 1_0 the unit of F. If $1_0 = 1$, then by Corollary 4.1.2.7 c),

$$x'(1) = y'(1) = \|y'\| = \|x'\| , \quad x' \in E'_+ .$$

So assume $1_0 \neq 1$. Then

$$(1 - 1_0)^2 = 1 - 1_0 - 1_0 + 1_0 = 1 - 1_0 ,$$

so that

$$\|1 - 1_0\| = 1 .$$

Take $\varphi \in]0, \frac{\pi}{2}]$ and $x \in E$ such that

$$\|x\| = \sin \varphi .$$

Then

$$\|(\cos \varphi)1_0 + x(1 - 1_0)\|^2 = \|((\cos \varphi)1_0 + x(1 - 1_0))((\cos \varphi)1_0 + (1 - 1_0)x^*)\| =$$
$$= \|(\cos \varphi)^2 1_0 + x(1 - 1_0)x^*\| \leq \cos^2 \varphi + \|x\|^2 = 1 .$$

By the hypothesis of b) and Corollary 4.1.2.7 c),

$$\|x'\| \geq x'((\cos \varphi)1_0 + x(1 - 1_0)) = x'(1_0) \cos \varphi + \langle x(1 - 1_0), x' \rangle =$$
$$= y'(1_0) \cos \varphi + \langle x, (1 - 1_0)x' \rangle = \|x'\| \cos \varphi + \langle x, (1 - 1_0)x' \rangle ,$$

so that

$$\langle x, (1 - 1_0)x' \rangle \leq (1 - \cos \varphi)\|x'\| = 2 \sin^2 \frac{\varphi}{2} \|x'\| .$$

Since x is arbitrary,

$$2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \|(1 - 1_0)x'\| = \sin \varphi \|(1 - 1_0)x'\| \leq 2 \sin^2 \frac{\varphi}{2} \|x'\|$$

$$\|(1 - 1_0)x'\| \leq \operatorname{tg} \frac{\varphi}{2} \|x'\| .$$

Since φ is also arbitrary,

$$\|(1 - 1_0)x'\| = 0 ,$$

$$(1 - 1_0)x' = 0 ,$$

$$0 = \langle 1, (1 - 1_0)x' \rangle = \langle 1 - 1_0, x' \rangle = x'(1) - x'(1_0) ,$$

$$x'(1) = x'(1_0) = y'(1_0) = \|y'\| = \|x'\|$$

by the hypothesis of b) and Corollary 4.1.2.7 c). By Corollary 4.1.2.7 c) again,

$$x' \in E'_+ .$$ ■

Corollary 4.1.2.9 *If T is a locally compact space and μ is a bounded Radon measure on T, then μ is positive iff*

$$\mu(T) = \|\mu\| .$$

If T is compact, then the assertion follows from Corollary 4.1.2.7 c). If T is not compact, let T^* be its Alexandroff compactification. Then μ may be identified canonically with an element of $\mathcal{C}(T^*)'$ and the assertion follows from the above remark (and Corollary 4.1.1.14). ■

Corollary 4.1.2.10 (0) *Let E be a Gelfand unital C^*-algebra and take $x \in E$. Then the following are equivalent:*

a) *x is not invertible.*

b) *x is a topological divisor of 0.*

a $\Rightarrow$ b. By Corollary 4.1.2.5, we may assume that $E = \mathcal{C}(T)$, where T is a compact space. There is a $t_0 \in T$, with $x(t_0) = 0$. For each $n \in \mathbb{N}$, put

$$x_n := e_T - e_T \wedge |nx|,$$

where $\wedge$ denotes the infimum. Then $x_n \in \mathcal{C}(T)$,

$$0 \leq x_n \leq e_T, \quad x_n(t_0) = 1, \quad \|x_n\| = 1,$$

$$|(xx_n)(t)| \leq \frac{1}{n}\|x\|$$

for every $n \in \mathbb{N}$ and $t \in T$. Hence

$$\|xx_n\| \leq \frac{1}{n}\|x\|$$

for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} xx_n = 0.$$

Hence x is a topological divisor of 0.

b $\Rightarrow$ a follows from Proposition 2.2.4.25. ■

Corollary 4.1.2.11 *Let F be the C^*-hull of the involutive commutative complex Banach algebra E. Let $u : E \to F$ be the canonical map and T the closed subspace*

$$\{x' \in \sigma(E) \mid x' \text{ is involutive}\}$$

of $\sigma(E)$. Then F is commutative, $u'(\sigma(F)) = T$, and the map

$$\sigma(F) \longrightarrow T, \quad y' \longmapsto y' \circ u = u'y'$$

is a homeomorphism.

F is commutative by Proposition 2.1.1.13 e). Take $y' \in \sigma(F)$. Then y' is involutive (Theorem 4.1.2.1, Proposition 2.4.2.3 a $\Rightarrow$ d), so that $u'y' \in T$, i.e. $u'(\sigma(F)) \subset T$. By Proposition 4.1.1.22 f), the map

$$\sigma(F) \longrightarrow T, \quad y' \longmapsto u'y'$$

is bijective. This map is obviously continuous and if $\sigma(F)$ is not compact then $u'y'$ converges to the Alexandroff point of T whenever y' converges to the Alexandroff point of $\sigma(F)$. Hence the above map is a homeomorphism. ■

Theorem 4.1.2.12 (0) (Rickart, 1946) *Let F be a C^*-subalgebra of the C^*-algebra E. Then*

$$\sigma_E(x) = \sigma_F(x)$$

for every $x \in F$.

We may assume that E and F are unital (Theorem 4.1.1.13). Assume that x is not invertible in F. We want to show that x is not invertible in E.

Case 1 F is a Gelfand C^*-algebra

By Corollary 4.1.2.10 a $\Rightarrow$ b, x is a topological divisor of 0 in F. Then x is a fortiori a topological divisor of 0 in E. Hence by Corollary 4.1.2.10 b $\Rightarrow$ a, x is not invertible in E.

Case 2 x self-normal

By Corollary 4.1.2.3, $F(x, 1)$ is a Gelfand C^*-algebra. Since x is not invertible in $F(x, 1)$, it is not invertible in E by the first case.

Case 3 The General Case

Assume that x is invertible in E. Then x^* is invertible in E as well (Proposition 2.3.1.14). Hence x^*x and xx^* are invertible in E (Corollary 2.1.2.6). By Case 2, x^*x and xx^* are invertible in F and so x is invertible in F (Corollary 2.1.2.6), which is a contradiction. Hence x is not invertible in E.

By the above result,

$$\sigma_F(x) \subset \sigma_E(x).$$

The reverse inclusion follows from Corollary 2.1.3.14 a). ■

Corollary 4.1.2.13 (0) *Let E be a (unital) C^*-algebra. Then the following are equivalent for every $x \in \operatorname{Sn} E$:*

a) x *is selfadjoint* (x *is unitary*).

b) $\sigma(x) \subset \mathbb{R}$ $\quad(\sigma(x) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\})$.

a $\Leftrightarrow$ b. We define

$$F := E(x) \quad (F := E(x, 1)).$$

Then F is a Gelfand C^*–algebra (Corollary 4.1.2.3) and

$$\sigma_E(x) = \sigma_F(x),$$

by Theorem 4.1.2.12. By Corollary 4.1.2.5 (and Corollary 4.1.2.6), we may assume that $F = \mathcal{C}(T)$, with T a locally compact (compact) space. But in this case the assertion is trivial. ■

Corollary 4.1.2.14 *Let E be a symmetric involutive complex Banach algebra, F a complex C^*–algebra, and $u : E \to F$ an algebra homomorphism. If $u(\operatorname{Re} E) \subset \operatorname{No} F$ (e.g. F is commutative), then u is involutive and continuous and $\|u\| \leq 1$.*

Take $x \in \operatorname{Re} E$. Then

$$\sigma_F(ux) \subset \sigma_E(x) \cup \{0\} \subset \mathbb{R}$$

(Corollary 2.1.3.12, Proposition 2.3.1.29), so $ux \in \operatorname{Re} F$ (Corollary 4.1.2.13 b $\Rightarrow$ a). By Corollary 2.3.1.23 b $\Rightarrow$ a, u is involutive. By Corollary 4.1.1.20, u is continuous and $\|u\| \leq 1$. ■

Proposition 4.1.2.15 (0) *Let E, F be Gelfand unital C^*–algebras and $u : E \to F$ a unital algebra homomorphism.*

a) *There is a unique map*

$$\varphi : \sigma(F) \longrightarrow \sigma(E),$$

such that

$$\widehat{ux}^{F} = \widehat{x}^{E} \circ \varphi$$

for every $x \in E$.

b) *u is continuous and involutive, $\|u\| = 1$, and*

$$y' \in \sigma(F) \Longrightarrow \varphi(y') = y' \circ u = u'y'.$$

c) *φ is continuous.*

d) *u is surjective (injective) iff φ is injective (surjective).*

e) *u is bijective iff φ is a homeomorphism.*

f) *If there is an $x \in E$, with $E = E(x,1)$ and $F = F(ux,1)$ and if we identify $\sigma(E)$ with $\sigma_E(x)$ and $\sigma(F)$ with $\sigma_F(ux)$ via the homeomorphisms $\overset{E}{\widehat{x}}$ and $\overset{F}{\widehat{ux}}$, respectively (Corollary 4.1.2.4), then*

$$\sigma(F) \subset \sigma(E)$$

and

$$\varphi(\alpha) = \alpha$$

for every $\alpha \in \sigma(F)$.

We denote by v and w the Gelfand transforms of E and F, respectively. Then v and w are isometries of unital C^*-algebras (Corollary 4.1.2.5), and so

$$u_0 := w \circ u \circ v^{-1} : \mathcal{C}(\sigma(E)) \longrightarrow \mathcal{C}(\sigma(F))$$

is a unital algebra homomorphism.

a),b),c),d),e) follow from Proposition 2.4.3.6 and Proposition 2.4.1.17 a),b).

f) We have that

$$\sigma(F) = \sigma_F(ux) \subset \sigma_E(x) = \sigma(E)$$

(Corollary 2.1.3.12). Take $\alpha \in \sigma_F(ux)$. Let $y' \in \sigma(F)$ such that

$$\overset{F}{\widehat{ux}}(y') = \alpha .$$

By a),

$$\alpha = \overset{E}{\widehat{x}}(\varphi(\alpha))$$

so that

$$\varphi(\alpha) = \alpha . \qquad \blacksquare$$

Corollary 4.1.2.16 (0) *Let F be a unital C^*-subalgebra of the unital C^*-algebra E.*

a) *There is a unique map*

$$\varphi : \sigma(E) \longrightarrow \sigma(F)\,,$$

such that

$$\overset{E}{\widehat{x}} = \overset{F}{\widehat{x}} \circ \varphi$$

for every $x \in F$.

b) $x' \in \sigma(E) \Rightarrow \varphi(x') = x'|F$.

c) φ *is continuous and surjective.*

d) φ *is bijective iff* $E = F$.

e) *Every character of* F *can be extended to a character of* E.

f) *If there is an* $x \in F$ *with* $F = F(x,1)$ *and if we identify* $\sigma(F)$ *with* $\sigma(x)$ *via the homeomorphism* $\overset{F}{\widehat{x}}$ *(Corollary 4.1.2.4), then* $\varphi = \overset{E}{\widehat{x}}$.

a),b),c), and d) follow from Proposition 4.1.2.15 a),b),c),d),e).
e) follows from b) and c).
f) By a),

$$\overset{E}{\widehat{x}}(x') = \overset{F}{\widehat{x}}(\varphi(x'))$$

for every $x' \in \sigma(E)$. Since we have identified $\sigma(x)$ with $\sigma(F)$ via $\overset{F}{\widehat{x}}$ we see that $\varphi = \overset{E}{\widehat{x}}$. ■

Corollary 4.1.2.17 (0) *Let* E *be a unital Gelfand algebra and* $\mathfrak{F}$ *an upward directed set of unital* C^**-subalgebras of* E *the union of which is dense in* E. *Define*

$$\varphi_{F,G} : \sigma(G) \longrightarrow \sigma(F)\,, \quad x' \longmapsto x'|F$$

for all unital C^**-subalgebras* F, G *of* E *with* $F \subset G$.

a) $(\sigma(F), \varphi_{F,G})_{\mathfrak{F}}$ *is a projective system of compact spaces such that* $\varphi_{F,G}$ *is surjective for all* $F, G \in \mathfrak{F}$, $F \subset G$; *call* T *its projective limit.*

b) $\varphi_{F,E} = \varphi_{F,G} \circ \varphi_{G,E}$ *for all* $F, G \in \mathfrak{F}$, $F \subset G$; *call* $\varphi : \sigma(E) \to T$ *the projective limit of* $(\varphi_{F,E})_{F \in \mathfrak{F}}$.

c) φ *is a homeomorphism.*

d) *If* $\sigma(F)$ *is totally disconnected for every* $F \in \mathfrak{F}$ *(e.g.* F *is finite-dimensional), then* $\sigma(E)$ *is also totally disconnected.*

a) follows from Corollary 4.1.2.16 a),b),c) (and Theorem 2.4.1.3 c)).

b) is obvious.

c) φ is continuous. By a), it is surjective. Let $x', y' \in \sigma(E)$ with $\varphi(x') = \varphi(y')$. Then $x' = y'$ on $\bigcup_{F \in \mathfrak{F}} F$. Since this set is dense in E, we get $x' = y'$ (Proposition 2.2.4.19). Hence φ is injective and so it is a homeomorphism.

d) follows from c). ■

Definition 4.1.2.18 (0) *The selfadjoint idempotent elements of a* C^**-algebra are called* ***orthogonal projections****. If* E *is a* C^**-algebra, then we denote by* $\Pr E$ *the set of orthogonal projections of* E.

If F is a C^*-subalgebra of E, then $\Pr F \subset \Pr E$.

The reason for the name "orthogonal projection" will appear in Proposition 5.3.2.8.

Example 4.1.2.19

$$\Pr \mathcal{L}(\mathbb{K}^2) =$$

$$= \{0,1\} \cup \left\{ \begin{bmatrix} \cos^2\varphi & e^{i\theta}\cos\varphi\sin\varphi \\ e^{-i\theta}\cos\varphi\sin\varphi & \sin^2\varphi \end{bmatrix} \;\middle|\; \begin{array}{l} \varphi, \theta \in \mathbb{R}, \\ (\frac{\theta}{\pi} \in \mathbb{Z} \text{ if } \mathbb{K} = \mathbb{R}) \end{array} \right\}.$$

If $p, q \in \Pr \mathcal{L}(\mathbb{K}^2) \backslash \{0\}$ *with* $pq = 0$, *then* $p + q = 1$. *If we identify* $\mathcal{L}(\mathbb{K}^2)$ *with* $\mathcal{L}(\mathbb{K}^2)'$ *as in Example 2.3.5.3, then* $\tau_0(\mathcal{L}(\mathbb{K}^2))$ *coincides with*

$$\Pr \mathcal{L}(\mathbb{K}^2) \backslash \{0,1\}.$$

Take $\alpha, \delta \in \mathbb{R}$ and $\beta \in \mathbb{K}$. Then we have

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix}^2 = \begin{bmatrix} \alpha^2 + |\beta|^2 & \beta(\alpha+\delta) \\ \overline{\beta}(\alpha+\delta) & \delta^2 + |\beta|^2 \end{bmatrix}.$$

In particular,

$$\begin{bmatrix} \cos^2\varphi & e^{i\theta}\cos\varphi\sin\varphi \\ e^{-i\theta}\cos\varphi\sin\varphi & sin^2\varphi \end{bmatrix}^2 = \begin{bmatrix} \cos^2\varphi & e^{i\theta}\cos\varphi\sin\varphi \\ e^{-i\theta}\cos\varphi\sin\varphi & sin^2\varphi \end{bmatrix}$$

for all $\varphi, \theta \in \mathbb{R}$. If

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix} \in \Pr \mathcal{L}(\mathbb{K}^2) \backslash \{0, 1\},$$

then

$$\begin{cases} \alpha^2 + |\beta|^2 = \alpha, & \beta(\alpha + \delta) = \beta \\ \delta^2 + |\beta|^2 = \delta, & \overline{\beta}(\alpha + \delta) = \overline{\beta} \end{cases},$$

so that

$$\alpha, \delta \in \mathbb{R}_+, \quad \alpha + \delta = 1, \quad |\beta|^2 = \alpha\delta.$$

Hence there are $\varphi, \theta \in \mathbb{R}$ (with $\frac{\theta}{\pi} \in \mathbb{Z}$ if $\mathbb{K} = \mathbb{R}$) with

$$\alpha = \cos^2 \varphi, \quad \delta = \sin^2 \varphi, \quad \beta = e^{i\theta} \cos \varphi \sin \varphi.$$

This proves the first assertion.

By the above considerations, there are $\varphi, \psi, \theta, \rho \in \mathbb{R}$ (with $\frac{\theta}{\pi}, \frac{\rho}{\pi} \in \mathbb{Z}$ if $\mathbb{K} = \mathbb{R}$), with

$$p = \begin{bmatrix} \cos \varphi \\ e^{-i\theta} \sin \varphi \end{bmatrix} [\cos \varphi \;\; e^{i\theta} \sin \varphi],$$

$$q = \begin{bmatrix} \cos \psi \\ e^{-i\rho} \sin \psi \end{bmatrix} [\cos \psi \;\; e^{i\rho} \sin \psi].$$

Suppose first that

$$\sin \varphi \neq 0, \quad \sin \psi \neq 0.$$

It follows from $pq = 0$ that

$$\cos \varphi \cos \psi + e^{i(\theta - \rho)} \sin \varphi \sin \psi = 0,$$

so either $\frac{\theta - \rho}{2\pi} \in \mathbb{Z}$ or $\frac{\theta - \rho - \pi}{2\pi} \in \mathbb{Z}$. In the former case,

$$\cos(\varphi - \psi) = 0$$

and in the latter,

$$\cos(\varphi + \psi) = 0.$$

We get $p + q = 1$ in both cases. The same is true if $\sin \varphi = 0$ or $\sin \psi = 0$.

The last assertion follows from Example 2.3.5.3. ■

Example 4.1.2.20 *Let E be a Gelfand C^*-subalgebra of $\mathcal{L}(\mathbb{K}^2)$ of dimension n. Then $n \leq 2$, E is generated by a family $(p_i)_{i\in\mathbb{N}_n}$ in $\Pr\mathcal{L}(\mathbb{K}^2)\backslash\{0\}$ with*

$$p_i p_j = 0$$

for all distinct $i, j \in \mathbb{N}_n$, and every character of E is the restriction to E of a pure state of $\mathcal{L}(\mathbb{K}^2)$. If $n = 2$, then the unit of $\mathcal{L}(\mathbb{K}^2)$ belongs to E.

By the Gelfand transform of E, $\sigma(E)$ contains exactly n points and there is a family $(p_i)_{i\in\mathbb{N}_n}$ in $\Pr\mathcal{L}(\mathbb{K}^2)\backslash\{0\}$ generating E, with

$$p_i p_j = 0$$

for all distinct $i, j \in \mathbb{N}_n$. By Example 4.1.2.19, $n \leq 2$ and every character of E is the restriction to E of a pure state of $\mathcal{L}(\mathbb{K}^2)$. The last assertion follows from Example 4.1.2.19. ■

Proposition 4.1.2.21 (0) *Let x be a self-normal element of a C^*-algebra. Then the following are equivalent:*

a) *x is an orthogonal projection.*

b) *x is idempotent.*

c) *$\sigma(x) \subset \{0, 1\}$.*

In this case $\|x\| \in \{0, 1\}$.

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c follows from Corollary 2.1.3.7 (and Corollary 2.1.3.2).

c $\Rightarrow$ a. By Corollary 4.1.2.13 b $\Rightarrow$ a , x is selfadjoint. But then $x^2 - x$ is selfadjoint and by Theorem 2.1.3.4 b),

$$\sigma(x^2 - x) \subset \{0\},$$

so that

$$r(x^2 - x) = 0.$$

By Theorem 4.1.1.16 b),

$$x^2 - x = 0.$$

Hence x is idempotent.

The last assertion follows from Theorem 4.1.1.16 b). ■

Corollary 4.1.2.22 (0) *If x is an element of a C^*-algebra E, then the following are equivalent:*

a) x^*x *is idempotent.*

b) xx^* *is idempotent.*

c) $xx^*x = x$.

a $\Rightarrow$ b. By Proposition 2.1.3.10 b),

$$\sigma(xx^*)\backslash\{0\} = \sigma(x^*x)\backslash\{0\}\,.$$

Hence xx^* is idempotent (Proposition 4.1.2.21 c $\Leftrightarrow$ b).

b $\Rightarrow$ c. We have

$$\|x - xx^*x\|^2 = \|(x - xx^*x)(x^* - x^*xx^*)\| =$$

$$= \|xx^* - xx^*xx^* - xx^*xx^* + xx^*xx^*xx^*\| = 0\,,$$

so that

$$x = xx^*x\,.$$

c $\Rightarrow$ a. We have

$$(x^*x)^2 = x^*xx^*x = x^*x\,.$$ ■

Proposition 4.1.2.23 (0) *If x, y are elements of a C^*-algebra and p is an orthogonal projection, then*

$$\|px(1-p) + (1-p)yp\| = \sup\{\|px(1-p)\|\,,\ \|(1-p)yp\|\}\,.$$

Put

$$u := (1-p)x^*px(1-p)\,, \quad v := py^*(1-p)yp\,.$$

Then u and v are selfadjoint and

$$uv = vu = 0\,.$$

Hence the C^*-subalgebra of E generated by $\{u, v\}$ is a Gelfand C^*-algebra (Corollary 4.1.2.3). By means of the Gelfand transform, we get

$$\|u + v\| = \sup\{\|u\|, \|v\|\}$$

(Corollary 4.1.2.5). We deduce

$$\|px(1-p)+(1-p)yp\|^2 =$$

$$= \|((1-p)x^*p+py^*(1-p))(px(1-p)+(1-p)yp)\| =$$

$$= \|(1-p)x^*px(1-p)+py^*(1-p)yp\| = \|u+v\| = \sup\{\|u\|,\|v\|\} =$$

$$= \sup\{\|px(1-p)\|^2\,,\ \|(1-p)yp\|^2\}\,. \qquad \blacksquare$$

Proposition 4.1.2.24 *If E is a unital C^*–algebra, then*

$$x^*(\Pr E)x = \Pr E$$

for every unitary element x of E.

Take $p \in \Pr E$. Then x^*px is selfadjoint and

$$(x^*px)^2 = x^*pxx^*px = x^*p^2x = x^*px\,,$$

so that $x^*px \in \Pr E$,

$$x^*(\Pr E)x \subset \Pr E\,,$$

$$x(\Pr E)x^* \subset \Pr E\,,$$

$$\Pr E = x^*x(\Pr E)x^*x \subset x^*(\Pr E)x \subset \Pr E\,,$$

$$x^*(\Pr E)x = \Pr E\,. \qquad \blacksquare$$

Proposition 4.1.2.25 *If E is a unital C^*–algebra and*

$$A := \mathrm{Re}\,E \cap \mathrm{Un}\,E\,,$$

then the map

$$\Pr E \longrightarrow A\,, \quad p \longmapsto 2p-1$$

is bijective and

$$A \longrightarrow \Pr E\,, \quad x \longmapsto \frac{1}{2}(x+1)$$

is its inverse.

Now $2p-1 \in \operatorname{Re} E$ and

$$(2p-1)^2 = 4p^2 - 4p + 1 = 1\,,$$

so that $2p-1 \in A$, for every $p \in \Pr E$. We have $\frac{1}{2}(x+1) \in \operatorname{Re} E$ and

$$\left(\frac{1}{2}(x+1)\right)^2 = \frac{1}{4}(x^2+2x+1) = \frac{1}{2}(x+1)\,,$$

so that $\frac{1}{2}(x+1) \in \Pr E$ for every $x \in A$. ■

Proposition 4.1.2.26 *Let E be a Gelfand C^*-algebra. If $\sigma(x)\backslash\{0\}$ is discrete for every $x \in E$, then $\sigma(E)$ is discrete.*

Take $x' \in \sigma(E)$. Assume that $\{x'\}$ is not open. We construct inductively a strictly decreasing sequence $(K_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, K_n is a compact open set of $\sigma(E)$ containing x'. Take $n \in \mathbb{N}$ and assume the sequence has been constructed up to $n-1$. Take $y' \in K_{n-1}\backslash\{x'\}$ $(K_0 := \sigma(E))$ and $f \in \mathcal{C}_0(\sigma(E))$ with

$$f(x') = 1\,, \quad f(y') = 0\,, \quad \operatorname{Supp} f \subset K_{n-1}\,.$$

Since the Gelfand transform is surjective, there is an $x \in E$ with $\widehat{x} = f$. By hypothesis, $\sigma(x)\backslash\{0\}$ is discrete. Then

$$K_n := \overset{-1}{f}(1)$$

is a compact open set of $\sigma(E)$ containing x' and strictly contained in K_n. This completes the inductive construction.

Let g denote the function on $\sigma(E)$ which takes the value $\frac{n-1}{n}$ on $K_{n-1}\backslash K_n$ for every $n \in \mathbb{N}$ and 1 on $\bigcap_{n\in\mathbb{N}} K_n$. Then $g \in \mathcal{C}_0(\sigma(E))$. Using the surjectivity of the Gelfand transform again, there is a $y \in E$ such that $\widehat{y} = g$. But then

$$\sigma(y) = \left\{ \frac{n-1}{n} \;\middle|\; n \in \mathbb{N} \right\} \cup \{1\}$$

and 1 is not an isolated point of $\sigma(y)$. This contradicts our assumption, so $\{x'\}$ must be open and hence $\sigma(E)$ is discrete. ■

Proposition 4.1.2.27 (0) *Let E be a complex (real) C^*-algebra without unit, $\mathfrak{F}$ an approximate unit of E, and $\mathbb{K}\times E$ the unital C^*-algebra associated to E.*

a) $\lim_{x,\mathfrak{F}} x'(x^*x)$ *exists for every* $x' \in E'$ $(x' \in \operatorname{Re} E')$. *For* $x' \in E'$ $(x' \in \operatorname{Re} E')$ *define*

$$x_1' : \mathbb{K} \times E \longrightarrow \mathbb{K}, \quad (\alpha, x) \longmapsto x'(x) + \alpha \lim_{y,\mathfrak{F}} x'(y^*y)$$

and for $y' \in (\mathbb{K} \times E)'$ *define*

$$y_0' : E \longrightarrow \mathbb{K}, \quad x \longmapsto y'(0, x).$$

b) $(\alpha, x) \in \mathbb{K} \times E \Rightarrow \frac{\|x\|}{2} \leq \|(\alpha, x)\| = \lim_{y,\mathfrak{F}} \|x + \alpha y\|, |\alpha| \leq \|(\alpha, x)\|$.

c) $x' \in E' \Rightarrow x_1' \in (\mathbb{K} \times E)', \|x'\| = \|x_1'\|, (x_1')^* = (x'^*)_1, x' = x_{10}'$.

d) *For every* $y' \in (\mathbb{K} \times E)'$ *and* $(\alpha, x) \in \mathbb{K} \times E$,

$$\langle (\alpha, x), y' - y_{01}' \rangle = \alpha(y'(1,0) - \lim_{y,\mathfrak{F}} y_0'(y^*y)),$$

$$\|y' - y_{01}'\| = |y'(1,0) - \lim_{y,\mathfrak{F}} y_0'(y^*y)|,$$

$$y_0' = y_{010}'.$$

e) $x' \in E_+' \Rightarrow x_1' \in (\mathbb{K} \times E)_+'$.

f) $y' \in (\mathbb{K} \times E)_+' \Rightarrow y_0' \in E_+', y_{01}', y' - y_{01}' \in (\mathbb{K} \times E)_+'$.

g) $y' \in \operatorname{Re}(\mathbb{K} \times E)' \Rightarrow \|y'\| = \|y_0'\| + \|y' - y_{01}'\|$.

h) $x' \in \operatorname{Re} E', y' \in \operatorname{Re}(\mathbb{K} \times E)', x' = y_0', \|x'\| = \|y'\| \Rightarrow y' = x_1'$.

i) *For every* $x' \in E'$ $(x' \in \operatorname{Re} E')$,

$$x' \in E_+' \Longleftrightarrow \lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|.$$

j) *The map*

$$E' \longrightarrow (\mathbb{K} \times E)', \quad x' \longmapsto x_1'$$

is an involutive homomorphism of E*-modules (where* E *is identified with* $\{0\} \times E$*).*

By Proposition 2.3.4.13 b), E' is an ordered Banach space and E'_+ is the set of continuous positive linear forms on E.

a) If x' is positive, then by Proposition 2.3.4.10 a)

$$\lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|\,.$$

If x' is selfadjoint, then by Corollary 4.1.2.7 d), it is a difference of two positive linear forms and the above limit exists in this case as well. If $\mathbb{K} = \mathbb{C}$, then the general case follows from the decomposition

$$x' = \operatorname{re} x' + i \operatorname{im} x'\,.$$

b) and c) Step 1 $\|(\alpha, x)\| = \lim_{y,\mathfrak{F}} \|x + \alpha y\|$

Take $y, z \in E^{\#}$. Then

$$\|xz + \alpha yz\| \leq \|x + \alpha y\|\,\|z\| \leq \|x + \alpha y\| \leq \|x - xy\| + \|xy + \alpha y\| =$$

$$= \|x - xy\| + \|(\alpha, x)(0, y)\| \leq \|x - xy\| + \|(\alpha, x)\|\,\|(0, y)\| \leq$$

$$\leq \|x - xy\| + \|(\alpha, x)\|\,,$$

so that

$$\|(\alpha, x)z\| = \|xz + \alpha z\| = \lim_{y,\mathfrak{F}} \|xz + \alpha zy\| \leq \liminf_{y,\mathfrak{F}} \|x + \alpha y\|$$

and

$$\|(\alpha, x)\| = \sup_{z \in E^{\#}} \|(\alpha, x)z\| \leq \liminf_{y,\mathfrak{F}} \|x + \alpha y\| \leq$$

$$\leq \limsup_{y,\mathfrak{F}} \|x + \alpha y\| \leq \limsup_{y,\mathfrak{F}} \|x - xy\| + \|(\alpha, x)\| \leq \|(\alpha, x)\|$$

(Theorem 4.1.1.13). Hence

$$\|(\alpha, x)\| = \lim_{y,\mathfrak{F}} \|x + \alpha y\|\,.$$

Step 2 $|\alpha| \leq \|(\alpha, x)\|$

Put

$$y' : \mathbb{K} \times E \longrightarrow \mathbb{K}\,, \quad (\beta, y) \longmapsto \beta\,.$$

Then $y' \in \tau(\mathbb{K} \times E)$ so that

$$|\alpha| = |y'(\alpha, x)| \leq \|(\alpha, x)\| .$$

Step 3 c)

$$|(x'(x) + \alpha x'(y^* y)| = |x'(x + \alpha y^* y)| \leq \|x'\| \, \|x + \alpha y^* y\|$$

for every $y \in E$. Thus

$$|x_1'(\alpha, x)| = \lim_{y, \mathfrak{F}} |x'(x) + \alpha x'(y^* y)| \leq \|x'\| \lim_{y, \mathfrak{F}} \|x + \alpha y^* y\| = \|x'\| \, \|(\alpha, x)\| ,$$

by Step 1 and Proposition 2.3.4.9. Hence x_1' is continuous and

$$\|x_1'\| \leq \|x'\| .$$

The reverse inequality and the relation

$$x' = x_{10}'$$

are trivial.

Given $(\alpha, y) \in \mathbb{K} \times E$,

$$\langle (\alpha, y), (x_1')^* \rangle = \overline{\langle (\overline{\alpha}, y^*), x_1' \rangle} = \overline{x'(y^*)} + \alpha \overline{\lim_{z, \mathfrak{F}} x'(z^* z)} =$$

$$= x'^*(y) + \alpha \lim_{z, \mathfrak{F}} x'^*(z^* z) = \langle (\alpha, y), (x'^*)_1 \rangle ,$$

so that

$$(x_1')^* = (x'^*)_1 .$$

Step 4 $\frac{\|x\|}{2} \leq \|(\alpha, x)\|$

Take $x' \in E'^{\#}$ with

$$\|x\| = x'(x) .$$

Then

$$x'(x) = x_1'(0, x) \leq |x_1'(0, x) + x_1'(\alpha, 0)| + |x_1'(\alpha, 0)| =$$

$$= |x_1'(\alpha, x)| + |\alpha| \, |\lim_{y, \mathfrak{F}} x'(y^* y)| \leq \|x_1'\| \, \|(\alpha, x)\| + |\alpha| \leq$$

$$\leq \|x'\| \, \|(\alpha, x)\| + \|(\alpha, x)\| = 2\|(\alpha, x)\|$$

(Steps 2 and 3). It follows that

$$\frac{\|x\|}{2} \leq \|(\alpha, x)\| \, .$$

d) We have that

$$\langle (\alpha, x), y' - y'_{01} \rangle = y'(\alpha, x) - y'(0, x) - \alpha \lim_{y,\mathfrak{F}} y'_0(y^* y) =$$

$$= \alpha (y'(1,0) - \lim_{y,\mathfrak{F}} y'_0(y^* y)) \, ,$$

so, by b),

$$|\langle (\alpha, x), y' - y'_{01} \rangle| = |\alpha| \, |y'(1,0) - \lim_{y,\mathfrak{F}} y'_0(y^* y)| \leq$$

$$\leq \|(\alpha, x)\| \, |y'(1,0) - \lim_{y,\mathfrak{F}} y'_0(y^* y)| \, .$$

Hence

$$\|y' - y'_{01}\| \leq |y'(1,0) - \lim_{y,\mathfrak{F}} y'_0(y^* y)| =$$

$$= |\langle (1,0), y' - y'_{01} \rangle| \leq \|y' - y'_{01}\| \, ,$$

$$\|y' - y'_{01}\| = |y'(1,0) - \lim_{y,\mathfrak{F}} y'_0(y^* y)| \, .$$

By the first equality,

$$y'_0 - y'_{010} = 0 \, .$$

e) If $(\alpha, x) \in \mathbb{K} \times E$, then

$$x'_1((\alpha, x)^*(\alpha, x)) = x'_1(|\alpha|^2, \overline{\alpha} x + \alpha x^* + x^* x) =$$

$$= x'(\overline{\alpha} x + \alpha x^* + x^* x) + |\alpha|^2 \lim_{y,\mathfrak{F}} x'(y^* y) =$$

$$= \lim_{y,\mathfrak{F}} x'(\overline{\alpha} y^* x + \alpha x^* y + x^* x + |\alpha|^2 y^* y) = \lim_{y,\mathfrak{F}} x'((\alpha y + x)^*(\alpha y + x)) \in \mathbb{R}_+$$

by Proposition 2.3.4.9, so that x'_1 is positive.

f) It is obvious that y'_0 is positive. By e), y'_{01} is positive as well. By d) and Corollary 2.3.4.7, if $(\alpha, x) \in \mathbb{K} \times E$, then

$$\langle (\alpha, x)^*(\alpha, x), y' - y'_{01} \rangle = \langle (|\alpha|^2, \overline{\alpha}x + \alpha x^* + x^* x), y' - y'_{01} \rangle =$$

$$= |\alpha|^2 (y'(1,0) - \lim_{y,\mathfrak{F}} y'_0(y^* y)) \geq$$

$$\geq |\alpha|^2 (\|y'\| - |\lim_{y,\mathfrak{F}} y'(0, y^* y)|) \geq 0,$$

so that $y' - y'_{01}$ is positive.

g) First suppose that y' is positive. By f), y'_{01} and $y' - y'_{01}$ are positive, so that

$$\|y'\| = \|y'_{01}\| + \|y' - y'_{01}\| = \|y'_0\| + \|y' - y'_{01}\|$$

by c) and Corollary 2.3.4.14.

Now let y' be arbitrary. By Corollary 4.1.2.7 d), there are $u', v' \in (\mathbb{K} \times E)'_+$, with

$$y' = u' - v', \quad \|y'\| = \|u'\| + \|v'\|.$$

Then

$$y'_{01} = u'_{01} - v'_{01},$$

and, by the above considerations,

$$\|y'\| \leq \|y' - y'_{01}\| + \|y'_{01}\| \leq$$

$$\leq \|u' - u'_{01}\| + \|v' - v'_{01}\| + \|u'_{01}\| + \|v'_{01}\| =$$

$$= \|u'\| + \|v'\| = \|y'\|,$$

$$\|y'\| = \|y' - y'_{01}\| + \|y'_{01}\|.$$

h) By g),

$$\|y' - y'_{01}\| = 0,$$

so that

$$y' = y'_{01} = x'_1 .$$

i) By c),e), and Corollary 4.1.2.7 c),

$$(x' \in E'_+) \Longleftrightarrow (x'_1 \in (\mathbb{K} \times E)'_+) \Longleftrightarrow (x'_1(1,0) = \|x'_1\|) \Longleftrightarrow (\lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|) .$$

j) Take $(x, x') \in E \times E'$. If $(\alpha, y) \in \mathbb{K} \times E$, then

$$\langle (\alpha, y), (0, x)x'_1 \rangle = \langle (\alpha, y)(0, x), x'_1 \rangle = \langle (0, \alpha x + yx), x'_1 \rangle =$$

$$= \langle \alpha x + yx, x' \rangle = \langle yx, x' \rangle + \alpha \lim_{z,\mathfrak{F}} \langle z^*zx, x' \rangle =$$

$$= \langle y, xx' \rangle + \alpha \lim_{z,\mathfrak{F}} \langle z^*z, xx' \rangle = \langle (\alpha, y), (xx')_1 \rangle$$

(Proposition 2.3.4.9), so that

$$(xx')_1 = (0, x)x'_1 .$$

Similarly,

$$(x'x)_1 = x'_1(0, x) .$$

Hence the map

$$E' \longrightarrow (\mathbb{K} \times E)' , \quad x' \longmapsto x'_1$$

is a homomorphism of E–modules. By c), this map is involutive. ■

Remark. Because of c) and j), we may identify E' with an E–submodule of $(\mathbb{K} \times E)'$. i) holds even for $\mathbb{K} = \mathbb{R}$ and $x' \in E'$ (Proposition 4.3.6.1 g)).

Example 4.1.2.28 (7) *Let E be a (unital) C^*–algebra, $(p_\iota)_{\iota \in I}$ an (infinite) family in* $\operatorname{Pr} E$ *such that*

$$p_\iota p_\lambda = \delta_{\iota\lambda} p_\iota$$

for all $\iota, \lambda \in I$, and F (resp. G) the (unital) C^*–subalgebra of E generated by $\{p_\iota \mid \iota \in I\}$. We endow I with the discrete topology (and denote by $\widetilde{I}$ the Alexandroff compactification of I).

a) *F and G are Gelfand C^*–algebras.*

b) *Given* $(\alpha_\iota)_{\iota\in I} \in c_0(I)$, $(\alpha_\iota p_\iota)_{\iota\in I}$ *is summable and the map*

$$c_0(I) \longrightarrow F, \quad (\alpha_\iota)_{\iota\in I} \longmapsto \sum_{\iota\in I} \alpha_\iota p_\iota$$

is an isomorphism of C^**-algebras.*

c) *The isomorphism of b) may be extended uniquely to an isomorphism* $c(I) \to G$ *of unital* C^**-algebras.*

d) $\sigma(F)$ *(resp.* $\sigma(G)$*) is homeomorphic to* I *(to* $\widetilde{I}$*).*

a) follows from Corollary 4.1.2.3.

b) follows from Proposition 4.1.1.25.

c) Let $\mathfrak{F}$ be the filter on I of cofinite subsets of I, i.e.

$$\mathfrak{F} := \{I\backslash A \mid A \in \mathfrak{P}_f(I)\}.$$

Given $\alpha \in c(I)$, put

$$\widetilde{\alpha} := \lim_{\iota,\mathfrak{F}} \alpha_\iota .$$

By b), the map

$$u : c(I) \longrightarrow G, \quad \alpha \longmapsto \widetilde{\alpha}1 + \sum_{\iota\in I}(\alpha_\iota - \widetilde{\alpha})p_\iota$$

is well defined and bijective. It is obviously linear, involutive, and extends the map of b). Take $\alpha, \beta \in c(I)$. Then

$$u(\alpha)u(\beta) = \left(\widetilde{\alpha}1 + \sum_{\iota\in I}(\alpha_\iota - \widetilde{\alpha})p_\iota\right)\left(\widetilde{\beta}1 + \sum_{\iota\in I}(\beta_\iota - \widetilde{\beta})p_\iota\right) =$$

$$= \widetilde{\alpha}\widetilde{\beta}1 + \sum_{\iota\in I}(\widetilde{\alpha}(\beta_\iota - \widetilde{\beta}) + (\alpha_\iota - \widetilde{\alpha})\widetilde{\beta} + (\alpha_\iota - \widetilde{\alpha})(\beta_\iota - \widetilde{\beta}))p_\iota =$$

$$= \widetilde{\alpha\beta}1 + \sum_{\iota\in I}(\widetilde{\alpha}\beta_\iota - \widetilde{\alpha}\widetilde{\beta} + \alpha_\iota\widetilde{\beta} - \widetilde{\alpha}\widetilde{\beta} + \alpha_\iota\beta_\iota - \alpha_\iota\widetilde{\beta} - \widetilde{\alpha}\beta_\iota + \widetilde{\alpha}\widetilde{\beta})p_\iota =$$

$$= \widetilde{\alpha\beta}1 + \sum_{\iota\in I}((\alpha\beta)_\iota - \widetilde{\alpha\beta})p_\iota = u(\alpha\beta).$$

Hence u is an algebra homomorphism. By Corollary 4.1.1.21, u is an isomorphism of C^*-algebras.

d) follows from b),c), Example 2.4.3.2, and Example 2.4.3.5. ■

Proposition 4.1.2.29 (7) *Let E be a unital C^*-algebra, F a unital C^*-subalgebra of E, and $(p_\iota)_{\iota\in I}$ a family in $\operatorname{Pr} E$ such that*

$$F \cup \{p_\iota \mid \iota \in I\} \subset \{p_\iota \mid \iota \in I\}^c$$

and such that

$$\prod_{\iota\in J} p_\iota \neq 0\,, \quad \sum_{\iota\in J} p_\iota \neq 1$$

for every $J \in \mathfrak{P}_f(I)$, $J \neq \emptyset$. Let further G and H be the unital C^*-subalgebras of E generated by $\{p_\iota \mid \iota \in I\}$ and $F \cup \{p_\iota \mid \iota \in I\}$, respectively.

a) *G is a Gelfand C^*-algebra and the map*

$$\sigma(G) \longrightarrow \{0,1\}^I\,, \quad x' \longmapsto (x'(p_\iota))_{\iota\in I}$$

is a homeomorphism.

b) *The unital C^*-algebra $\mathcal{C}(\sigma(G), F)$ of continuous maps of $\sigma(G)$ into F (Example 4.1.1.6) is isomorphic to H.*

c) *The following are equivalent for every $(\alpha_\iota)_{\iota\in I} \in \mathbb{K}^I$:*

c_1) $(\alpha_\iota)_{\iota\in I} \in \ell^1(I)$.

c_2) $\beta \in \mathbb{K} \Rightarrow (\alpha_\iota(\beta 1 + p_\iota))_{\iota\in I}$ *is summable.*

c_3) $(\alpha_\iota(1 + p_\iota))_{\iota\in I}$ *is summable.*

c_4) $(\alpha_\iota p_\iota)_{\iota\in I}$ *is summable.*

If these conditions are fulfilled, $\beta \in \mathbb{R}_+$, and $(\alpha_\iota)_{\iota\in I} \in \mathbb{R}_+^I$ then:

c_5) $\|\sum_{\iota\in I} \alpha_\iota(\beta 1 + p_\iota)\| = (1+\beta)\sum_{\iota\in I} \alpha_\iota$.

Take $J \in \mathfrak{P}_f(I)$. We denote by G_J and H_J the unital C^*-subalgebra of E generated by $\{p_\iota \mid \iota \in I\}$ and $F \cup \{p_\iota \mid \iota \in I\}$, respectively, and set

$$p_{J,K} := \left(\prod_{\iota\in K} p_\iota\right)\left(\prod_{\iota\in J\setminus K} (1 - p_\iota)\right)$$

for all $K \subset J$, where

$$\prod_{\iota\in\emptyset} p_\iota = \prod_{\iota\in\emptyset} (1 - p_\iota) = 1\,.$$

Then $p_{J,K} \in \Pr E$ for all $K \subset J$,

$$p_{J,K} p_{J,L} = 0$$

for all distinct $K, L \in \mathfrak{P}(J)$, and

$$\sum_{K \subset J} p_{J,K} = 1 .$$

a) Let $J \in \mathfrak{P}_f(I)$. Then

$$G_J = \left\{ \sum_{K \subset J} \alpha_K p_{J,K} \;\middle|\; (\alpha_K)_{K \subset J} \in \mathbb{K}^{\mathfrak{P}(J)} \right\} .$$

Hence G_J is a Gelfand C^*-algebra and the map

$$\sigma(G_J) \longrightarrow \{0,1\}^J , \quad x' \longmapsto (x'(p_\iota))_{\iota \in J}$$

is a homeomorphism.

$(G_J)_{J \in \mathfrak{P}_f(I)}$ is an upward directed family of unital C^*-subalgebras of G the union of which is dense in G. Hence G is a Gelfand C^*-algebra and by Corollary 4.1.2.17 c), the map

$$\sigma(G) \longrightarrow \{0,1\}^I , \quad x' \longmapsto (x'(p_\iota))_{\iota \in I}$$

is a homeomorphism.

b) Let $J \in \mathfrak{P}_f(I)$. The map

$$F^{\mathfrak{P}(J)} \longrightarrow H_J , \quad (x_K)_{K \subset J} \longmapsto \sum_{K \subset J} x_K p_{J,K}$$

is an isometry of unital C^*-algebras (Corollary 4.1.1.21). We identify H_J with $F^{\mathfrak{P}(J)}$ via this map and set

$$\widehat{x} : \sigma(G) \longrightarrow F , \quad x' \longmapsto \sum_{K \subset J} x'(p_{J,K}) x_K$$

for every $x \in H_J$. $(p_{J,K} F)_{K \subset J}$ is a family of C^*-subalgebras of E, H_J is the C^*-subalgebra of E generated by $\bigcup_{K \subset J} p_{J,K} F$, and

$$(p_{J,K} F)(p_{J,L} F) = \{0\}$$

whenever $K, L \in \mathfrak{P}(J)$ are distinct. By Proposition 4.1.1.25,

$$\|\widehat{x}\| = \sup_{K \subset J} \|x_K\| = \|x\|$$

for every $x \in H_J$.

Denote by $\mathcal{C}(\sigma(G), F)$ the unital C^*–algebra of continuous maps of $\sigma(G)$ into F. By the above considerations, the map

$$\bigcup_{J \in \mathfrak{P}_f(I)} H_J \longrightarrow \mathcal{C}(\sigma(G), F), \quad x \longmapsto \widehat{x}$$

is an involutive unital algebra homomorphism preserving the norms. Since $\bigcup_{J \in \mathfrak{P}_f(I)} H_J$ is dense in H, the above map may be extended to a homomorphism of C^*–algebras

$$u : H \longrightarrow \mathcal{C}(\sigma(G), F)$$

which preserves the norms. We have to show that u is surjective.

Take $y \in \mathcal{C}(\sigma(G), F)$ and $\varepsilon > 0$. Since y is uniformly continuous, there is, by a), a $J \in \mathfrak{P}_f(I)$ such that for all $K \subset J$ and

$$x', y' \in \overset{-1}{p}_{J,K}(1)$$

we have

$$\|y(x') - y(y')\| < \varepsilon .$$

Given $K \subset J$, take $x'_K \in \overset{-1}{p}_{J,K}(1)$. We set

$$x := \sum_{K \subset J} y(x'_K) p_{J,K} .$$

Then $x \in H_J$ and

$$\|\widehat{x} - y\| < \varepsilon .$$

Since ε is arbitrary and since $u(H)$ is closed, we see that $y \in u(H)$. Hence u is surjective.

$c_1 \Rightarrow c_2, c_2 \Rightarrow c_3$, and $c_2 \Rightarrow c_4$ are trivial.

$c_3 \Rightarrow c_1, c_4 \Rightarrow c_1$, and $c_2 \Rightarrow c_5$. By a), there is an $x' \in \sigma(G)$ such that $x'(p_\iota) = 1$ for all $\iota \in I$. By Proposition 1.2.1.16, $(\alpha_\iota)_{\iota \in I}$ is summable. Moreover, if $(\alpha_\iota)_{\iota \in I} \in \mathbb{R}_+^I$ and $\beta \in \mathbb{R}_+$ then

$$(1+\beta) \sum_{\iota \in I} \alpha_\iota = \sum_{\iota \in I} x'(\alpha_\iota(\beta 1 + p_\iota)) = x' \left(\sum_{\iota \in I} \alpha_\iota(\beta 1 + p_\iota) \right) \leq$$

$$\leq \left\| \sum_{\iota \in I} \alpha_\iota(\beta 1 + p_\iota) \right\| \leq \sum_{\iota \in I} \|\alpha_\iota(\beta 1 + p_\iota)\| \leq (1+\beta) \sum_{\iota \in I} \alpha_\iota ,$$

$$\left\| \sum_{\iota \in I} \alpha_\iota(\beta 1 + p_\iota) \right\| = (1+\beta) \sum_{\iota \in I} \alpha_\iota . \qquad \blacksquare$$

4.1.3 Functional calculus in C^*–Algebras

Lemma 4.1.3.0 (0) *Let K be a compact set of $\mathbb{K}$ such that $K\backslash\{0\}$ is discrete. Given $P \in \mathbb{K}[t]$, define*

$$\widetilde{P} : K \longrightarrow \mathbb{K}, \quad \alpha \longmapsto P(\alpha).$$

Put

$$\mathcal{F} := \{\widetilde{P} \mid P \in \mathbb{K}[t]\},$$

$$\mathcal{G} := \{\widetilde{P} \mid P \in \mathbb{K}[t], (0 \in K \Longrightarrow \widetilde{P}(0) = 0)\}.$$

Then $\mathcal{F}$ is dense in $\mathcal{C}(K)$ and $\mathcal{G}$ is dense in $\{x \in \mathcal{C}(K) \mid 0 \in K \Rightarrow x(0) = 0\}$.

Take $\alpha \in K\backslash\{0\}$. Put

$$A := \{\beta \in K \mid |\beta| \geq |\alpha|, \beta \neq \alpha\}$$

$$x : K \longrightarrow \mathbb{K}, \quad \beta \longmapsto \delta_{\alpha,\beta},$$

and

$$P_n(t) := \left(\frac{t}{\alpha}\right)^n \frac{\prod\limits_{\beta \in A}(t-\beta)}{\prod\limits_{\beta \in A}(\alpha-\beta)}$$

for every $n \in \mathbb{N}$. Then $\widetilde{P_n} \in \mathcal{G}$, $\widetilde{P_n}(\alpha) = 1$, $\widetilde{P_n} = 0$ on A for every $n \in \mathbb{N}$, and $(\widetilde{P_n})_{n\in\mathbb{N}}$ converges uniformly to x. Hence $x \in \widetilde{\mathcal{G}}$. The assertions now follow easily. ■

Theorem 4.1.3.1 (0) (Gelfand–Naimark, 1943) *Let x be a self-normal element of a unital C^*-algebra E. Put $F := E(x, 1)$,*

$$j : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha,$$

and let v be the Gelfand transform of F.

a) *There is a unique involutive unital algebra homomorphism*

$$u : \mathcal{C}(\sigma(x)) \longrightarrow E,$$

with

$$uj = x.$$

This u is continuous. For $f \in \mathcal{C}(\sigma(x))$ put

$$f(x) := uf.$$

b) $u(\mathcal{C}(\sigma(x))) = F$ *and the map*

$$\mathcal{C}(\sigma(x)) \longrightarrow F\,, \quad f \longmapsto f(x)$$

is an isometry of unital C^*-algebras. In particular, $f(x)$ is self-normal for every $f \in \mathcal{C}(\sigma(x))$.

c) $f \in \mathcal{C}(\sigma(x)) \Rightarrow f(x) = v^{-1}\left(f \circ \overset{F}{\widehat{x}}\right)$, $\overset{F}{\widehat{f(x)}} = f \circ \overset{F}{\widehat{x}}$.

d) $E(x) = \{f(x) \mid f \in \mathcal{C}(\sigma(x))\,, (0 \in \sigma(x) \Rightarrow f(0) = 0)\}$.

e) *If $\sigma(x)\backslash\{0\}$ is discrete, then*

$$\{f(x) \mid f \in \mathcal{C}(\sigma(x))\,, (0 \in \sigma(x) \Longrightarrow f(0) = 0)\}\,,$$

$$(\{f(x) \mid f \in \mathcal{C}(\sigma(x))\})$$

is the closed (unital) subalgebra of E generated by x.

f) *Let F be an E-module and $a \in F$ such that*

$$ax = ax^* = 0 \quad (xa = x^*a = 0)\,.$$

Then

$$af(x) = 0 \quad (f(x)a = 0)$$

for every $f \in \mathcal{C}(\sigma(x))$ with

$$0 \in \sigma(x) \Longrightarrow f(0) = 0\,.$$

a) F is a Gelfand C^*-algebra (Corollary 4.1.2.3), v is an isometry of unital C^*-algebras (Corollary 4.1.2.5), and

$$\sigma_E(y) = \sigma_F(y) = \overset{F}{\widehat{y}}(\sigma(F))$$

for every $y \in F$ (Theorem 4.1.2.12, Corollary 2.4.1.7 a)).

We define

$$u : \mathcal{C}(\sigma(x)) \longrightarrow E\,, \quad f \longmapsto v^{-1}\left(f \circ \overset{F}{\widehat{x}}\right)\,.$$

Then u is a continuous involutive unital algebra homomorphism with

$$uj = x\,.$$

Let

$$w : \mathcal{C}(\sigma(x)) \longrightarrow E$$

be an involutive unital algebra homomorphism such that

$$wj = x\,.$$

Given $P \in \mathbb{K}[s,t]$, put

$$\widetilde{P} : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto P(\alpha, \overline{\alpha})$$

and

$$\mathcal{F} := \{\widetilde{P} \mid P \in \mathbb{K}[s,t]\}\,.$$

Then $\mathcal{F}$ is an involutive unital subalgebra of $\mathcal{C}(\sigma(x))$ which separates the points of $\sigma(x)$. By the Weierstrass–Stone Theorem, $\mathcal{F}$ is a dense set of $\mathcal{C}(\sigma(x))$. We have that

$$u\widetilde{P} = P(u(j), u(\overline{j})) = P(u(j), u(j)^*) = P(x, x^*) =$$

$$= P(w(j), w(j)^*) = P(w(j), w(\overline{j})) = w\widetilde{P}$$

for every $P \in \mathbb{K}[s,t]$. Thus u and w coincide on $\mathcal{F}$. Since they are continuous (Corollary 4.1.1.20) and $\mathcal{F}$ is dense in $\mathcal{C}(\sigma(x))$, they coincide.

b) and c). Since the map

$$\sigma(F) \longrightarrow \sigma(x)\,, \quad x' \longmapsto \overset{F}{\widehat{x}}(x')$$

is a homeomorphism (Corollary 4.1.2.4), the map

$$\mathcal{C}(\sigma(x)) \longrightarrow \mathcal{C}(\sigma(F))\,, \quad f \longmapsto f \circ \overset{F}{\widehat{x}}$$

is an isometry of unital C^*–algebras. Hence the map

$$\mathcal{C}(\sigma(x)) \longrightarrow F\,, \quad f \longmapsto v^{-1}(f \circ \overset{F}{\widehat{x}}) = f(x)$$

is also an isometry of unital C^*–algebra. Given $f \in \mathcal{C}(\sigma(x))$, $f(x)$ is self–normal since F is a Gelfand C^*–algebra.

d) We put

$$\mathcal{F}_0 := \{f \in \mathcal{F} \mid 0 \in \sigma(x) \Longrightarrow f(0) = 0\},$$

where $\mathcal{F}$ is the set defined in the proof of a). By Corollaries 1.3.5.15 and 1.3.5.16, $\mathcal{F}_0$ is a dense set of

$$\{f \in \mathcal{C}(\sigma(x)) \mid 0 \in \sigma(x) \Longrightarrow f(0) = 0\}.$$

Given $f \in \mathcal{F}_0$, $f(x) \in E(x)$ and so

$$\{f(x) \mid f \in \mathcal{C}(\sigma(x)),\ (0 \in \sigma(x) \Longrightarrow f(0) = 0)\} \subset E(x),$$

by a). On the other hand, $\{f(x) \mid f \in \mathcal{F}_0\}$ is a dense set of $E(x)$, so that

$$\{f(x) \mid f \in \mathcal{C}(\sigma(x)),\ (0 \in \sigma(x) \Longrightarrow f(0) = 0)\} = E(x),$$

by b).

e) follows from b) and Lemma 4.1.3.0.

f) If $\mathcal{F}_0$ is the set defined in the proof of d) then

$$af(x) = 0 \quad (f(x)a = 0)$$

for every $f \in \mathcal{F}_0$. It follows

$$af(x) = 0 \quad (f(x)a = 0)$$

for every $f \in \mathcal{C}(\sigma(x))$ with

$$0 \in \sigma(x) \Longrightarrow f(0) = 0. \qquad \blacksquare$$

Remark. Let F, G be unital C^*–subalgebras of the unital C^*–algebra E. Let x be a self–normal element of E belonging to $F \cap G$ and take $f \in \mathcal{C}(\sigma(x))$. Then

$$E(x,1) = F(x,1) = G(x,1)$$

and so the element $f(x)$ defined in the above theorem by means of E, F, and G coincide. In particular, if E is a real C^*–algebra, $x \in \operatorname{Re} E$, $f \in \mathcal{C}(\sigma(x))$, and $\overset{\circ}{E}$ denotes the complexification of E, then $f(x)$ defined with respect to E on $\overset{\circ}{E}$ coincide.

Definition 4.1.3.2 (0) *Let E be a C^*–algebra without unit. Let F be the unital C^*–algebra associated to E. Take a self–normal element x of E and take $f \in \mathcal{C}(\sigma(x))$. Let $f(x)$ denote the element of F defined in Theorem 4.1.3.1. If $f(0) = 0$, then $f(x) \in E$ (Theorem 4.1.3.1 d)).*

Corollary 4.1.3.3 (0) *Let E be a C^*-algebra. Take $x \in \operatorname{Re} E$ and put*

$$\mathcal{F} := \{f \in \mathcal{C}(\sigma(x)) \mid f(\sigma(x)) \subset \mathbb{R}\},$$

$$\mathcal{G} := \{f \in \mathcal{F} \mid 0 \in \sigma(x) \Longrightarrow f(0) = 0\}.$$

a) *$\{f(x) \mid f \in \mathcal{G}\}$ is the closed real subalgebra of E generated by x and it is contained in $\operatorname{Re} E$.*

b) *If E is unital, then $\{f(x) \mid f \in \mathcal{F}\}$ is the closed unital real subalgebra of E generated by x and it is contained in $\operatorname{Re} E$.*

By Corollary 4.1.2.13 a $\Rightarrow$ b , $\sigma(x) \subset \mathbb{R}$.

a) (resp. b)). Let F be the closed (unital) real subalgebra of E generated by x. Since $\{f(x) \mid f \in \mathcal{G}\ (f \in \mathcal{F})\}$ is a closed (unital) real subalgebra of E containing x, it contains F. Take $f \in \mathcal{G}\ (f \in \mathcal{F})$. By Corollaries 1.3.5.15 and 1.3.5.16, there is a sequence $(P_n)_{n\in\mathbb{N}}$ in $\mathbb{R}[t]$ converging uniformly to f on $\sigma(x)$. If $0 \notin \sigma(x)$ or if $0 \in \sigma(x)$ and $f(0) = 0$, we may additionally assume that $P_n(0) = 0$ for each $n \in \mathbb{N}$. Thus, by Theorem 4.1.3.1 b),

$$f(x) = \lim_{n\to\infty} P_n(x) \in F \cap \operatorname{Re} E.$$

Hence

$$F = \{f(x) \mid f \in \mathcal{G}\ (f \in \mathcal{F})\} \subset \operatorname{Re} E. \qquad \blacksquare$$

Corollary 4.1.3.4 (0) *Let E, F be C^*-algebras and $u : E \to F$ an involutive algebra homomorphism. Take $x \in \operatorname{Sn} E$ and $f \in \mathcal{C}(\sigma(x))$ such that*

$$0 \in \sigma(x) \Longrightarrow f(0) = 0.$$

Then

$$uf(x) = f(ux).$$

If E, F, and u are unital, then we may drop the hypothesis that

$$0 \in \sigma(x) \Longrightarrow f(0) = 0.$$

If u is injective, then

$$\|uy\| = \|y\|$$

for every $y \in E$.

Since

$$u(E(x)) \subset F(ux)\,, \quad (u(E(x,1)) \subset F(ux,1))$$

(Proposition 2.3.1.20, Corollary 2.3.2.13, Corollary 4.1.1.20), we may assume that

$$E = E(x)\,, \quad F = F(ux)\,, \quad (E = E(x,1)\,, \quad F = F(ux,1))$$

(Theorem 4.1.3.1 b)). Then E, F are Gelfand C^*-algebras (Corollary 4.1.2.3) and the assertion follows from Proposition 4.1.2.15 a),f) and Theorem 4.1.3.1 c),d).

We now prove the last assertion. First assume that y is self-normal and take $f \in \mathcal{C}(\sigma_E(y))$ with

$$f(0) = 0$$

if $0 \in \sigma_E(y)$. Then

$$\sigma_F(uy) \subset \sigma_E(y) \cup \{0\}$$

(Corollary 2.1.3.12). From

$$uf(y) = f(uy)$$

and the injectivity of u we get

$$f(uy) = 0 \Longrightarrow f(y) = 0\,.$$

It follows that

$$\sigma_E(y) \subset \sigma_F(uy)$$

and that

$$\|uy\| = \|y\|$$

(Proposition 2.3.1.16, Theorem 4.1.1.16 b)).

Now let y be arbitrary. Then

$$\|uy\|^2 = \|(uy)^*(uy)\| = \|u(y^*y)\| = \|y^*y\| = \|y\|^2\,,$$

$$\|uy\| = \|y\|\,. \qquad \blacksquare$$

Corollary 4.1.3.5 (0) *Let F be a Gelfand unital C^*-subalgebra of the unital C^*-algebra E. Take $x \in F$ and $f \in C(\sigma(x))$. Then*

$$\overset{F}{\widehat{f(x)}} = f \circ \overset{F}{\widehat{x}}, \quad \sigma(f(x)) = f(\sigma(x)),$$

$$f(x) \in \operatorname{Re} E \Longleftrightarrow f \text{ real},$$

$$f(x) \in \operatorname{Un} E \Longleftrightarrow f(\sigma(x)) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}.$$

Given $y \in E(x,1)$, define

$$\widetilde{y} : \sigma(E(x,1)) \longrightarrow \mathbb{K}, \quad y' \longmapsto y'(y).$$

There is a continuous map

$$\varphi : \sigma(F) \longrightarrow \sigma(E(x,1)),$$

such that

$$\overset{F}{\widehat{y}} = \widetilde{y} \circ \varphi$$

for every $y \in E(x,1)$ (Corollary 4.1.2.16 a),c)). By Theorem 4.1.3.1 a),b),c)

$$\overset{F}{\widehat{f(x)}} = \widetilde{f(x)} \circ \varphi = (f \circ \widetilde{x}) \circ \varphi = f \circ \overset{F}{\widehat{x}},$$

so that

$$\sigma(f(x)) = \overset{F}{\widehat{f(x)}}(\sigma(F)) = f\left(\overset{F}{\widehat{x}}(\sigma(F))\right) = f(\sigma(x))$$

(Corollary 2.4.1.7 a)). We deduce that

$$(f(x) \in \operatorname{Re} E) \Longleftrightarrow (\sigma(f(x)) \subset \mathbb{R}) \Longleftrightarrow (f(\sigma(x)) \subset \mathbb{R}) \Longleftrightarrow (f \text{ real}),$$

$$(f(x) \in \operatorname{Un} E) \Longleftrightarrow (\sigma(f(x)) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}) \Longleftrightarrow$$

$$\Longleftrightarrow (f(\sigma(x)) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\})$$

(Corollary 4.1.2.13). ■

Corollary 4.1.3.6 *Let x be a normal element of a complex C^*-algebra and define*

$$f : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \alpha \longmapsto \operatorname{re}\alpha \quad (\textit{resp.}\ \operatorname{im}\alpha)\,.$$

Then

$$f(x) = \operatorname{re} x \quad (\textit{resp.}\ f(x) = \operatorname{im} x)$$

and

$$\sigma(\operatorname{re} x) = \{\operatorname{re}\alpha \mid \alpha \in \sigma(x)\}\,,$$

$$\sigma(\operatorname{im} x) = \{\operatorname{im}\alpha \mid \alpha \in \sigma(x)\}\,.$$

Put

$$P := \frac{1}{2}(s+t) \in \mathbb{C}[s,t] \quad \left(\text{resp. } P := \frac{1}{2i}(s-t) \in \mathbb{C}[s,t]\right)\,.$$

Then

$$f(\alpha) = P(\alpha, \overline{\alpha})$$

for every $\alpha \in \sigma(x)$, so that

$$f(x) = P(x, x^*) = \operatorname{re} x \quad (\text{resp. } \operatorname{im} x)$$

(Theorem 4.1.3.1 a)). By Corollary 4.1.3.5,

$$\sigma(\operatorname{re} x) = \{\operatorname{re}\alpha \mid \alpha \in \sigma(x)\}\,,$$

$$\sigma(\operatorname{im} x) = \{\operatorname{im}\alpha \mid \alpha \in \sigma(x)\}\,. \qquad \blacksquare$$

Corollary 4.1.3.7 (0) (Russo-Dye, 1966) *Every element of a unital complex C^*-algebra may be represented as a linear combination of four unitary elements.*

It is sufficient to show that every selfadjoint element may be represented as a linear combination of two unitary elements (Proposition 2.3.1.22). So let x be a selfadjoint element of a unital complex C^*-algebra. We may assume that $\|x\| \leq 1$. Then

$$\sigma(x) \subset [-1,1]$$

(Corollary 4.1.2.13, Theorem 4.1.1.16 b)). Put

$$f : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \alpha \longmapsto \alpha + i\sqrt{1-\alpha^2}\,.$$

By Corollary 4.1.2.13 and Corollary 4.1.3.5, $f(x)$ and $\overline{f}(x)$ are unitary. We have

$$x = \frac{1}{2}(f(x) + \overline{f}(x))$$

(Theorem 4.1.3.1 a)). ■

Corollary 4.1.3.8 (0) *Let E be a unital C^*–algebra. Take $x \in \operatorname{Sn} E$ and $P, Q \in \mathbb{K}[s,t]$ such that $Q(x,x^*)$ is invertible and define*

$$f : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \frac{P(\alpha,\overline{\alpha})}{Q(\alpha,\overline{\alpha})}\,.$$

Then

$$f(x) = P(x,x^*)Q(x,x^*)^{-1}\,.$$

We put

$$g : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto P(\alpha,\overline{\alpha})\,,$$

$$h : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto Q(\alpha,\overline{\alpha})\,.$$

Then

$$g(x) = P(x,x^*)\,, \quad h(x) = Q(x,x^*)$$

(Theorem 4.1.3.1 a)). By Corollary 4.1.3.5,

$$0 \notin \sigma(h(x)) = h(\sigma(x))\,.$$

Hence h does not vanish on $\sigma(x)$ and f is well–defined. We have that

$$g = fh\,,$$

so that

$$P(x,x^*) = g(x) = f(x)h(x) = f(x)Q(x,x^*)$$

(Theorem 4.1.3.1 a)) and

$$f(x) = P(x,x^*)Q(x,x^*)^{-1}\,.$$ ■

Corollary 4.1.3.9 (0) *Let E be a unital C^*-algebra and take $x \in \operatorname{Sn} E$. Let $\sum_{n=0}^{\infty} \alpha_n t^n$ be a power series in $\mathbb{K}$ with radius of convergence strictly greater than $\|x\|$ and define*

$$f : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \sum_{n=0}^{\infty} \alpha_n \alpha^n .$$

Then

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n$$

(Proposition 2.2.3.1, Proposition 2.2.3.2).

We define

$$P_n(t) = \sum_{k=0}^{n} \alpha_k t^k \in \mathbb{K}[t] ,$$

$$f_n : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto P_n(\alpha)$$

for every $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \|f_n - f\| = 0 ,$$

and so

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \alpha_n x^n$$

(Theorem 4.1.3.1 a),b)). ■

Corollary 4.1.3.10 (0) *If x is a normal element of a unital complex C^*-algebra, then e^{ix} is unitary iff x is selfadjoint.*

We define

$$f : \sigma(x) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto e^{i\alpha} .$$

Then

$$e^{ix} = f(x)$$

(Corollary 4.1.3.9) and

$$\sigma(f(x)) = f(\sigma(x))$$

(Corollary 4.1.3.5). Thus, by Corollary 4.1.2.13,

$$(e^{ix} \in \operatorname{Un} E) \Longleftrightarrow (f(\sigma(x)) \subset \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}) \Longleftrightarrow (\sigma(x) \subset \mathbb{R}) \Longleftrightarrow$$

$$\Longleftrightarrow (x \text{ selfadjoint}) .$$ ■

Corollary 4.1.3.11 (0) *Let E be a unital C^*-algebra. Take $x \in \operatorname{Sn} E$, $f \in \mathcal{C}(\sigma(x))$, and $g \in \mathcal{C}(\sigma(f(x)))$. Then*

$$g(f(x)) = g \circ f(x) .$$

$f(x)$ is self-normal and

$$\sigma(f(x)) = f(\sigma(x))$$

(Theorem 4.1.3.1 b), Corollary 4.1.3.5), so that $g(f(x))$ and $g \circ f$ are well-defined. We have

$$\widehat{g \circ f(x)} = (g \circ f) \circ \widehat{x} = g \circ (f \circ \widehat{x}) = g \circ \widehat{f(x)} = \widehat{g(f(x))} ,$$

where the Gelfand transform is taken with respect to $E(x, 1)$ (Corollary 4.1.3.5). Thus

$$g \circ f(x) = g(f(x))$$

(Corollary 4.1.2.5). ■

Corollary 4.1.3.12 *Let x be a unitary element of a unital complex C^*-algebra E. If*

$$\sigma(x) \neq \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$$

then there is a $y \in \operatorname{Re} E$ such that $e^{iy} = x$.

There is a $\theta_0 \in \mathbb{R}$, with

$$\sigma(x) \subset \{e^{i\theta} \mid \theta_0 < \theta < \theta_0 + 2\pi\}$$

(Corollary 4.1.2.13 a $\Rightarrow$ b). We define

$$f : \sigma(x) \longrightarrow]\theta_0, \theta_0 + 2\pi[, \quad e^{i\theta} \longmapsto \theta ,$$

$$y := f(x) .$$

By Corollary 4.1.3.5, y is selfadjoint. Since for each $\alpha \in \sigma(x)$,

$$e^{if(\alpha)} = \alpha$$

we see that

$$x = e^{if(x)} = e^{iy}$$

(Corollary 4.1.3.9, Corollary 4.1.3.11). ■

Remark. The condition

$$\sigma(x) \neq \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$$

is not necessary but it cannot be dropped. Indeed, if $E := \mathcal{C}(\mathbb{T})$ and

$$x : \mathbb{T} \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \alpha,$$

then $x \neq e^{iy}$ for every $y \in \operatorname{Re} E$. On the other hand, if $E := \mathcal{C}([0, 2\pi])$ and

$$y : [0, 2\pi] \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \alpha,$$

then $y \in \operatorname{Re} E$, e^{iy} is unitary, and

$$\sigma(e^{iy}) = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}.$$

Corollary 4.1.3.13 *Take $n \in \mathbb{N}$. If x is a unitary element of $\mathcal{L}(\mathbb{C}^n)$ (Example 4.1.1.7), then there is a $y \in \operatorname{Re} \mathcal{L}(\mathbb{C}^n)$ with $e^{iy} = x$.*

The assertion follows from Corollary 4.1.3.12, since $\sigma(x)$ contains at most n points. ■

Proposition 4.1.3.14 (0) *Let x be a self-normal element of a unital C^*-algebra and take $P \in \mathbb{K}[s, t]$. Then*

$$P(x, x^*) = 0$$

iff

$$\sigma(x) \subset \{\alpha \in \mathbb{K} \mid P(\alpha, \overline{\alpha}) = 0\}.$$

We define

$$f : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto P(\alpha, \overline{\alpha}).$$

Then $f(x)$ is self-normal (Theorem 4.1.3.1 b)) and

$$\sigma(P(x, x^*)) = \sigma(f(x)) = f(\sigma(x))$$

(Theorem 4.1.3.1 a), Corollary 4.1.3.5). By Theorem 4.1.1.16 c),

$$(P(x, x^*) = 0) \Longleftrightarrow (\sigma(P(x, x^*)) = \{0\}) \Longleftrightarrow (\sigma(x) \subset \{\alpha \in \mathbb{K} \mid P(\alpha, \overline{\alpha}) = 0\}).$$

■

Corollary 4.1.3.15 (0) *If x is a self-normal element of a unital C^*-algebra and $\alpha \in \mathbb{K}$, then*

$$x = \alpha 1$$

iff

$$\sigma(x) = \{\alpha\}\,.$$

Define

$$P := t - \alpha \in \mathbb{K}[t]\,.$$

Then

$$x = \alpha 1 \Longleftrightarrow P(x) = 0 \Longleftrightarrow \sigma(x) = \{\alpha\}$$

by Proposition 4.1.3.14. ■

Corollary 4.1.3.16 *If x is a self-normal element of a unital C^*-algebra, then*

$$e^x = 1$$

iff

$$\sigma(x) \subset 2\pi i\mathbb{Z}\,.$$

Define

$$f : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto e^\alpha\,.$$

Then $e^x = f(x)$ (Corollary 4.1.3.9),

$$\sigma(e^x) = \sigma(f(x)) = f(\sigma(x))$$

(Corollary 4.1.3.5), and $f(x)$ is self-normal (Theorem 4.1.3.1 b)). Hence

$$e^x = 1 \Longleftrightarrow \sigma(e^x) = \{1\} \Longleftrightarrow \sigma(x) \subset 2\pi i\mathbb{Z}$$

by Corollary 4.1.3.15. ■

Proposition 4.1.3.17 (6) *Let x be a normal element of a unital complex C^*-algebra, α an isolated singularity of the resolvent of x, and*

$$f : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \beta \longmapsto \delta_{\alpha\beta}\,,$$

$$g : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \beta \longmapsto \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \beta \neq \alpha \\ 0 & \text{if } \beta = \alpha\,. \end{cases}$$

Then

$$\frac{1}{t-\alpha} f(x) - \sum_{n=0}^{\infty} (t-\alpha)^n g(x)^{n+1}$$

is the Laurent series of the resolvent of x at α,

$$(\alpha 1 - x) f(x) = 0\,,$$

and

$$g(x)(\alpha 1 - x) = f(x) - 1\,.$$

In particular, α is a pole of order 1 and the residue is an orthogonal projection.

For some $r > 0$,

$$U := U_r^{\mathbb{C}}(\alpha) \backslash \{\alpha\} \subset \mathbb{C} \backslash \sigma(x)\,.$$

Take $t \in U$. Define

$$f' : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \beta \longmapsto \frac{f(\beta)}{t-\beta}\,,$$

$$g' : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \beta \longmapsto \frac{1 - f(\beta)}{t-\beta}\,.$$

Then

$$f'(\beta) = \frac{f(\beta)}{t-\alpha}\,,$$

$$g'(\beta) = \frac{1 - f(\beta)}{(t-\alpha) - (\beta-\alpha)} = -\sum_{n=0}^{\infty} (t-\alpha)^n \frac{1 - f(\beta)}{(\beta-\alpha)^{n+1}} = -\sum_{n=0}^{\infty} (t-\alpha)^n g(\beta)^{n+1}\,,$$

$$f'(\beta) + g'(\beta) = \frac{1}{t-\beta}$$

for every $\beta \in \sigma(x)$, so that

$$(t1 - x)^{-1} = f'(x) + g'(x) = \frac{1}{t-\alpha} f(x) - \sum_{n=0}^{\infty} (t-\alpha)^n g(x)^{n+1}$$

by Theorem 4.1.3.1 a), and Corollary 4.1.3.9. Hence α is a pole of order 1 and the residue is an orthogonal projection (Theorem 4.1.3.1 a),b), Proposition 4.1.2.21 b $\Rightarrow$ a). The equalities

$$(\alpha 1 - x) f(x) = 0\,, \quad g(x)(\alpha 1 - x) = f(x) - 1$$

follow by the functional calculus. ■

Corollary 4.1.3.18 *Let x be a normal element of a unital complex C^*–algebra and α, β distinct isolated singularities of the resolvent of x. Let y, z be the residues of the resolvent of x at α and β, respectively. Then*

$$yz = 0\,.$$

Define

$$f : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \gamma \longmapsto \delta_{\alpha\gamma}\,,$$

$$g : \sigma(x) \longrightarrow \mathbb{C}\,, \quad \gamma \longmapsto \delta_{\beta\gamma}\,.$$

By Proposition 4.1.3.17,

$$y = f(x)\,, \quad z = g(x)\,,$$

so that

$$yz = f(x)g(x) = (fg)(x) = 0$$

(Theorem 4.1.3.1 a)). ■

Example 4.1.3.19 *Let*

$$x := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

be a self-normal element of $\mathcal{L}(\mathbb{K}^2)$ (Example 4.1.1.7) and take $f \in C(\sigma(x))$. Put

$$\rho := \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}\,,$$

$$a := \frac{\alpha+\delta+\rho}{2}, \quad b := \frac{\alpha+\delta-\rho}{2},$$

and assume $\rho \neq 0$. *Then* $\sigma(x) = \{a, b\}$,

$$f(x) = \frac{f(a)}{\rho}\begin{bmatrix} \frac{\alpha-\delta+\rho}{2} & \beta \\ \gamma & \frac{\delta-\alpha+\rho}{2} \end{bmatrix} - \frac{f(b)}{\rho}\begin{bmatrix} \frac{\alpha-\delta-\rho}{2} & \beta \\ \gamma & \frac{\delta-\alpha-\rho}{2} \end{bmatrix} =$$

$$= \frac{f(a)-f(b)}{\rho}\begin{bmatrix} \frac{\alpha-\delta}{2} & \beta \\ \gamma & \frac{\delta-\alpha}{2} \end{bmatrix} + \frac{f(a)+f(b)}{2}1 = \frac{f(a)-f(b)}{\rho}x + \frac{af(b)-bf(a)}{\rho}1,$$

and if $\mathbb{K} = \mathbb{C}$, *then the residue of the resolvent of* x *at* a *is*

$$\frac{1}{\rho}\begin{bmatrix} \frac{\alpha-\delta+\rho}{2} & \beta \\ \gamma & \frac{\delta-\alpha+\rho}{2} \end{bmatrix}.$$

It is obvious that $\sigma(x) = \{a, b\}$. Moreover, given $c \in \{a, b\}$,

$$\frac{f(a)-f(b)}{\rho}c + \frac{af(b)-bf(a)}{\rho} = f(c),$$

so that

$$f(x) = \frac{f(a)-f(b)}{\rho}\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} + \frac{af(b)-bf(a)}{\rho}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$= \frac{f(a)}{\rho}\begin{bmatrix} \alpha-b & \beta \\ \gamma & \delta-b \end{bmatrix} + \frac{f(b)}{\rho}\begin{bmatrix} a-\alpha & -\beta \\ -\gamma & a-\delta \end{bmatrix} =$$

$$= \frac{f(a)}{\rho}\begin{bmatrix} \frac{\alpha-\delta+\rho}{2} & \beta \\ \gamma & \frac{\delta-\alpha+\rho}{2} \end{bmatrix} + \frac{f(b)}{\rho}\begin{bmatrix} \frac{\delta-\alpha+\rho}{2} & -\beta \\ -\gamma & \frac{\alpha-\delta+\rho}{2} \end{bmatrix} =$$

$$= \frac{f(a)-f(b)}{\rho}\begin{bmatrix} \frac{\alpha-\delta}{2} & \beta \\ \gamma & \frac{\delta-\alpha}{2} \end{bmatrix} + \frac{f(a)+f(b)}{2}1.$$

The last assertion follows immediately from the previous one and Proposition 4.1.3.17. ■

Proposition 4.1.3.20 (0) *Let* E *be a unital* C^**–algebra. Take* $A \subset \mathbb{K}$ *and let* $f : A \to \mathbb{K}$ *be a continuous map. Put*

$$B := \{x \in \operatorname{Sn} E \mid \sigma(x) \subset A\}.$$

Then the map

$$B \longrightarrow E, \quad x \longmapsto f(x)$$

is continuous.

Take $x \in B$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in B which converges to x. Put

$$K := \sigma(x) \cup \left(\bigcup_{n\in\mathbb{N}} \sigma(x_n) \right) .$$

By Corollary 2.2.5.3, K is compact.

Let $\varepsilon > 0$. By the Weierstrass–Stone Theorem, there is a $P \in \mathbb{K}[s,t]$, with

$$\sup_{\alpha\in K} |f(\alpha) - P(\alpha,\overline{\alpha})| < \varepsilon .$$

Then

$$\|f(x) - f(x_n)\| \leq \|f(x) - P(x,x^*)\| +$$

$$+\|P(x,x^*) - P(x_n,x_n^*)\| + \|P(x_n,x_n^*) - f(x_n)\| <$$

$$< 2\varepsilon + \|P(x,x^*) - P(x_n,x_n^*)\|$$

for every $n \in \mathbb{N}$ (Theorem 4.1.3.1 a),b)), so that

$$\limsup_{n\to\infty} \|f(x) - f(x_n)| \leq 2\varepsilon .$$

Since ε is arbitrary, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$. ■

Corollary 4.1.3.21 *Let E be a unital C^*-algebra. Take $x \in \operatorname{Sn} E$ and let U an open set of $\mathbb{K}$ with*

$$U \cap \sigma(x) \neq \emptyset .$$

Then there is an $\varepsilon > 0$ such that

$$U \cap \sigma(y) \neq \emptyset$$

for every $y \in \operatorname{Sn} E$ with $\|x - y\| < \varepsilon$.

Take $f \in \mathcal{C}(\mathbb{K})$ with

$$\operatorname{Supp} f \subset U , \quad \|f|\sigma(x)\| \neq 0 .$$

Then

$$f(x) \neq 0$$

(Theorem 4.1.3.1 b)). By Proposition 4.1.3.20, there is an $\varepsilon > 0$, such that

$$f(y) \neq 0$$

for every $y \in \operatorname{Sn} E$ with $\|x - y\| < \varepsilon$. Thus

$$\sigma(y) \cap \operatorname{Supp} f \neq \emptyset$$

and so

$$U \cap \sigma(y) \neq \emptyset$$

for every $y \in \operatorname{Sn} E$ with $\|x - y\| < \varepsilon$. ■

Proposition 4.1.3.22 (0) *Let E be the C^*-direct sum of the family $(E_\iota)_{\iota \in I}$ of C^*-algebras (Example 4.1.1.6). Take $x \in \operatorname{Sn} E$ and $f \in \mathcal{C}(\sigma(x))$. Then*

$$\sigma_E(x) = \overline{\bigcup_{\iota \in I} \sigma_{E_\iota}(x_\iota)}$$

and

$$f(x)_\iota = f(x_\iota)$$

for every $\iota \in I$.

First observe that $x_\iota \in \operatorname{Sn} E_\iota$ for every $\iota \in I$. Take

$$\alpha \in \mathbb{K} \backslash \overline{\bigcup_{\iota \in I} \sigma_{E_\iota}(x_\iota)}.$$

Then there is an $\varepsilon > 0$ such that

$$U_\varepsilon^{\mathbb{K}}(\alpha) \cap \overline{\bigcup_{\iota \in I} \sigma_{E_\iota}(x_\iota)} = \emptyset.$$

Thus

$$\|(\alpha 1 - x_\iota)^{-1}\| = \|\alpha 1 - x_\iota\|^{-1} \leq \frac{1}{\varepsilon}$$

for every $\iota \in I$ (Theorem 4.1.3.1 b)). Hence

$$\alpha \notin \sigma_E(x),$$

$$\sigma_E(x) \subset \overline{\bigcup_{\iota \in I} \sigma_{E_\iota}(x_\iota)}.$$

The reverse inclusion is trivial.

The relation

$$\iota \in I \Longrightarrow P(x, x^*)_\iota = P(x_\iota, x_\iota^*)$$

holds for every $P \in \mathbb{K}[s,t]$. By continuity,

$$f(x)_\iota = f(x_\iota)$$

for every $\iota \in I$. ■

Proposition 4.1.3.23 (0) *Let E be a real C^*-algebra. Take $x \in \operatorname{Re} E$ and $f \in \mathcal{C}(\sigma(x))$. If $(x,0)$ denotes the element of the complexification $\overset{\circ}{E}$ of E, then*

$$f((x,0)) = (f(x),0)\,.$$

We first remark that by Proposition 4.1.1.15 (and Corollary 2.1.5.14),

$$\sigma_{\overset{\circ}{E}}((x,0)) = \sigma_E(x)\,,$$

so that $f((x,0))$ is well-defined. The assertion is trivial if f is a polynomial (Corollary 4.1.3.8). Since the set of polynomials is dense in $\mathcal{C}(\sigma(x))$, the assertion also holds for arbitrary f. ■

4.1.4 The Theorem of Fuglede–Putnam

Theorem 4.1.4.1 (0) (Fuglede, 1950, Putnam, 1951) *Let F be a unital E–module over the unital C^*–algebra E. Take $x, y \in \operatorname{Sn} E$ and $a \in F$ with*

$$xa = ay\,.$$

Then

$$f(x)a = af(y)$$

for every $f \in \mathcal{C}(\sigma(x) \cup \sigma(y))$. If $F = E$ then $x^*a = ay^*$.

Step 1 $\quad \mathbb{K} = \mathbb{C} \Rightarrow x^*a = ay^*$

By complete induction,

$$x^n a = ay^n$$

for every $n \in \mathbb{N}$. Hence

$$e^{i\alpha x}a = \sum_{n=0}^{\infty} \frac{i^n\alpha^n}{n!}x^n a = \sum_{n=0}^{\infty} \frac{i^n\alpha^n}{n!}ay^n = ae^{i\alpha y}\,,$$

$$a = e^{-i\overline{\alpha}x}e^{i\overline{\alpha}x}a = e^{-i\overline{\alpha}x}ae^{i\overline{\alpha}y}$$

for every $\alpha \in \mathbb{C}$. Define

$$f : \mathbb{C} \longrightarrow F\,, \quad \alpha \longmapsto e^{-i\alpha x^*}ae^{i\alpha y^*}\,.$$

Then f is analytic (Proposition 1.3.10.3, Corollary 1.3.10,10). Since $\overline{\alpha}x + \alpha x^*$ and $\overline{\alpha}y + \alpha y^*$ are selfadjoint, $e^{-i(\overline{\alpha}x+\alpha x^*)}$ and $e^{i(\overline{\alpha}y+\alpha y^*)}$ are unitary (Corollary 4.1.3.10), so that

$$\|e^{-i(\overline{\alpha}x+\alpha x^*)}\| = \|e^{i(\overline{\alpha}x+\alpha y^*)}\| = 1$$

for every $\alpha \in \mathbb{C}$ (Proposition 4.1.1.4). Hence

$$\|f(\alpha)\| = \|e^{-i\alpha x^*}ae^{i\alpha y^*}\| = \|e^{-i\alpha x^*}e^{-i\overline{\alpha}x}ae^{i\overline{\alpha}y}e^{i\alpha y^*}\| =$$

$$= \|e^{-i(\overline{\alpha}x+\alpha x^*)}ae^{i(\overline{\alpha}y+\alpha y^*)}\| \leq \|e^{-i(\overline{\alpha}x+\alpha x^*)}\|\,\|a\|\,\|e^{i(\overline{\alpha}y+\alpha y^*)}\| = \|a\|$$

for every $\alpha \in \mathbb{C}$ (Proposition 2.2.3.7 a)). Thus f is bounded and therefore, by Liouville's Theorem, it is constant. Thus

$$a = f(0) = f(\alpha) = a + \alpha(-ix^*a + iay^*) + \dots$$

for every $\alpha \in \mathbb{C}$ (Proposition 1.2.9.6), so that

$$-ix^*a + iay^* = 0$$

(Corollary 1.3.3.10) and

$$x^*a = ay^* .$$

Step 2 $f(x)a = af(y)$

If $\mathbb{K} = \mathbb{R}$, then

$$P(x)a = aP(y)$$

for all $P \in \mathbb{R}[t]$. If $\mathbb{K} = \mathbb{C}$, then, by Step 1, it follows that

$$P(x, x^*)a = aP(y, y^*)$$

for all $P \in \mathbb{C}[s, t]$. By the Weierstrass–Stone Theorem,

$$f(x)a = af(y) .$$

Step 3 The Last Assertion

The assertion follows from Step 1 by complexification. ■

Remark. a) The above ingenious proof of the first step is due to *M. Rosenblum (1958).*

b) Take $g \in \mathcal{C}(\sigma(x) \cup \sigma(y))$ with $f = g$ on $\sigma(x) \cap \sigma(y)$. Then, it is easy to see that

$$g(x)a = f(x)a = af(y) = ag(y) .$$

Corollary 4.1.4.2 (0) *Let A be a subset of a C^*-algebra E such that*

$$A \subset A^* \cup \operatorname{Sn} E ,$$

and G the C^*-subalgebra of E generated by A. Let F be an E-module.

a) *Take* $a \in F$ *with* $xa = ax$ *for every* $x \in A$. *Then* $xa = ax$ *for every* $x \in G$.

b) A^c *and* A^{cc} *are* C^**-subalgebras of* E *and* $A^c = G^c$.

c) *If* A *is commutative (and contained in* $\mathrm{Re}\, E$ *if* $\mathbb{K} = \mathbb{R}$*), then* A^{cc} *is commutative,* G *is a Gelfand* C^**-algebra, and* A *is contained in a maximal Gelfand* C^**-subalgebra of* E.

d) *Every self-normal element of* E *is contained in a maximal Gelfand* C^**-subalgebra of* E.

a) Define

$$H := \{x \in E \mid xa = ax\}.$$

Then H is obviously a closed subalgebra of E.

Take $x \in A$. If $x \in A^*$, then $x^* \in A$, and so $x^* \in H$. If $x \in \mathrm{Sn}\, E$, then

$$x^* a = a x^*,$$

by the Fuglede–Putnam Theorem, so that $x^* \in H$. Hence $A \cup A^* \subset H$ and H contains the subalgebra of E generated by $A \cup A^*$, i.e. the involutive subalgebra of E generated by A (Proposition 2.3.1.19). Finally, H contains G (Corollary 2.3.2.14 a)).

b) By a) and Corollary 2.2.1.7, A^c and A^{cc} are C^*–subalgebras of E. Thus

$$A \subset G \subset A^{cc},$$

and

$$A^c = A^{ccc} \subset G^c \subset A^c,$$

$$A^c = G^c$$

by Proposition 2.1.1.17 b),d).

c) By Proposition 2.1.1.17 e), A^{cc} is commutative, and so, by b), G is a Gelfand C^*–algebra (Corollary 4.1.2.3). The last assertion follows from the last assertion of Corollary 4.1.2.3.

d) follows from c). ■

Corollary 4.1.4.3 *Let P (resp. Q) be a polynomial in two (resp. five) variables. Let A be a subset of $\mathbb{K}$ with*

$$\alpha, \beta \in A \Longrightarrow P(\alpha, \beta) \in A\,,$$

and $f : A \to \mathbb{K}$ a continuous function with

$$\alpha, \beta \in A \Longrightarrow Q(\alpha, \beta, f(\alpha), f(\beta), f(P(\alpha, \beta))) = 0\,.$$

Further let x, y be self-normal elements of a unital C^*-algebra such that

$$xy = yx\,, \quad \sigma(x) \cup \sigma(y) \subset A\,.$$

Then

$$Q(x, y, f(x), f(y), f(P(x, y))) = 0\,.$$

Let F be the C^*–subalgebra generated by $\{x, y\}$. F is a Gelfand C^*–algebra (Corollary 4.1.4.2 c)). For $z \in F$ define

$$\widehat{z} : \sigma(F) \longrightarrow \mathbb{K}\,, \quad x' \longmapsto x'(z)\,.$$

Then

$$\sigma(P(x, y)) = \widehat{P(x, y)}(\sigma(F)) = (P(\widehat{x}, \widehat{y}))(\sigma(F)) \subset A$$

(Corollary 2.4.1.7 a), Proposition 2.4.1.2). Hence $f(P(x, y))$ is well–defined. Furthermore

$$\widehat{Q(x, y, f(x), f(y), f(P(x, y)))} = Q\left(\widehat{x}, \widehat{y}, \widehat{f(x)}, \widehat{f(y)}, \widehat{f(P(x, y))}\right) =$$

$$= Q(\widehat{x}, \widehat{y}, f \circ \widehat{x}, f \circ \widehat{y}, f \circ P(\widehat{x}, \widehat{y})) = 0$$

(Proposition 2.4.1.2, Corollary 4.1.3.5), so that

$$Q(x, y, f(x), f(y), f(P(x, y))) = 0$$

(Corollary 4.1.2.5). ■

Corollary 4.1.4.4 (0) *If x is a self-normal element of a unital C^*-algebra, then*

$$\{x\}^c \subset \{f(x)\}^c$$

for every $f \in \mathcal{C}(\sigma(x))$.

Let E be the unital C^*–subalgebra generated by x. Then $f(x) \in E$ (Theorem 4.1.3.1 b)), and so

$$\{x\}^c = E^c \subset \{f(x)\}^c$$

(Corollary 4.1.4.2 b)). ■

Proposition 4.1.4.5 (0) *Let E be a unital C^*-algebra. Let A, B subsets of $\mathbb{K}$ and $f : A \to B$ a homeomorphism. Put*

$$A_0 := \{x \in \operatorname{Sn} E \mid \sigma(x) \subset A\}, \quad B_0 := \{x \in \operatorname{Sn} E \mid \sigma(x) \subset B\}.$$

Then

$$\{f(x) \mid x \in A_0\} = B_0,$$

the map

$$A_0 \longrightarrow B_0, \quad x \longmapsto f(x)$$

is a homeomorphism, and

$$f^{-1}(f(x)) = x, \quad \{x\}^c = \{f(x)\}^c, \quad E(x,1) = E(f(x),1)$$

for every $x \in A_0$. If in addition

$$0 \in A \Longrightarrow f(0) = 0,$$

then

$$E(x) = E(f(x)).$$

Take $x \in A_0$. Then $f(x)$ is self-normal (Theorem 4.1.3.1 b)),

$$\sigma(f(x)) = f(\sigma(x)) \subset B$$

(Corollary 4.1.3.5), and

$$f^{-1}(f(x)) = x$$

(Corollary 4.1.3.11). It follows from the last equality that

$$\{x\}^c = \{f(x)\}^c, \quad E(x,1) = E(f(x),1)$$

(Corollary 4.1.4.4, Theorem 4.1.3.1 d)). By Proposition 4.1.3.20, the maps

$$A_0 \longrightarrow B_0, \quad x \longmapsto f(x),$$

$$B_0 \longrightarrow A_0, \quad x \longmapsto f^{-1}(x)$$

are continuous and so they are homeomorphisms. ■

Theorem 4.1.4.6 (0) *Let E be a unital C^*-algebra. Take $x \in E$ and let A be a subset of $\mathbb{K}$ which only has isolated points. Then the following are equivalent:*

a) *x is self-normal,*

$$\sigma(x) \subset A \cup \{0\},$$

and $0 \in A$ if it is an isolated point of $\sigma(x)$.

b) *There is a family $(x_\alpha)_{\alpha \in A}$ in $\Pr E$ such that:*

1) $x = \sum_{\alpha \in A} \alpha x_\alpha$,
2) $\alpha, \beta \in A \Rightarrow x_\alpha x_\beta = \delta_{\alpha\beta} x_\alpha$,
3) $\sigma(x)$ *finite* $\Rightarrow \sum_{\alpha \in A} x_\alpha = 1$.

If these assertions hold, then:

c) *Given $\alpha \in A$, define*

$$f_\alpha : \sigma(x) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \delta_{\alpha\beta}.$$

Then

$$f_\alpha(x) = x_\alpha.$$

In particular, the family $(x_\alpha)_{\alpha \in A}$ from b) is unique and

$$\sigma(x)\backslash\{0\} = \{\alpha \in A\backslash\{0\} \mid x_\alpha \neq 0\}.$$

d) *x and $\{x_\alpha \mid \alpha \in A\}$ generate the same closed subalgebra of E and $E(x,1)$ is the unital C^*-subalgebra of E generated by $\{x_\alpha \mid \alpha \in A\}$. It is finite dimensional iff $\sigma(x)$ is finite.*

e) *Given any $f \in C(\sigma(x))$ such that $f(0) = 0$ if $\sigma(x)$ is infinite, we have that*

$$f(x) = \sum_{\alpha \in \sigma(x)} f(\alpha) x_\alpha.$$

a $\Rightarrow$ b. For $\alpha \in A$, let f_α be the function defined in c) and put

$$x_\alpha := f_\alpha(x)$$

(f_α is continuous since α is an isolated point of A). By Corollary 4.1.3.5

$$\sigma(x_\alpha) = \sigma(f_\alpha(x)) = f_\alpha(\sigma(x)) \subset \{0,1\},$$

so that $x_\alpha \in \Pr E$ for every $\alpha \in A$ (Proposition 4.1.2.21 c $\Rightarrow$ a). We have $f_\alpha = 0$, so that $x_\alpha = 0$ for every $\alpha \in A \backslash \sigma(x)$ and

$$x_\alpha x_\beta = f_\alpha(x) f_\beta(x) = (f_\alpha f_\beta)(x) = \delta_{\alpha\beta} f_\alpha(x) = \delta_{\alpha\beta} x_\alpha$$

for all $\alpha, \beta \in A$ (Theorem 4.1.3.1 a)).

Take $\varepsilon > 0$ and

$$B := \{\alpha \in \sigma(x) \mid |\alpha| \geq \varepsilon\}.$$

Since $\sigma(x)$ is compact (Corollary 2.2.4.5) and A consists of isolated points only, B is a finite subset of A. Let C be a finite subset of A containing B and put

$$f : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha.$$

Then

$$x - \sum_{\alpha \in C} \alpha x_\alpha = f(x) - \sum_{\alpha \in C} \alpha f_\alpha(x) = \left(f - \sum_{\alpha \in C} \alpha f_\alpha \right)(x),$$

so that

$$\left\| x - \sum_{\alpha \in C} \alpha x_\alpha \right\| = \left\| f - \sum_{\alpha \in C} \alpha f_\alpha \right\| < \varepsilon$$

(Theorem 4.1.3.1 a),b)). Since ε is arbitrary, we deduce that

$$\sum_{\alpha \in A} \alpha x_\alpha = x.$$

If $\sigma(x)$ is finite, then

$$\sum_{\alpha \in A} f_\alpha = 1,$$

so that

$$\sum_{\alpha \in A} x_\alpha = \sum_{\alpha \in A} f_\alpha(x) = \left(\sum_{\alpha \in A} f_\alpha \right)(x) = 1.$$

b $\Rightarrow$ a & c & d & e. Let F be the unital C^*–subalgebra of E generated by $\{x_\alpha \mid \alpha \in A\}$. F is a Gelfand C^*–algebra and $x \in F$ (1), 2), and Corollary 4.1.4.2 c)). Hence x is self–normal. For $y \in F$ put

$$\widehat{y} : \sigma(F) \longrightarrow \mathbb{K}\,, \quad y' \longmapsto y'(y)$$

and for $\alpha \in A$ put

$$A_\alpha := \{t \in \sigma(F) \mid \widehat{x}_\alpha(t) = 1\}\,.$$

Then

$$\widehat{x}_\alpha \widehat{x}_\beta = \widehat{x_\alpha x_\beta} = \delta_{\alpha\beta}\widehat{x}_\alpha$$

for all $\alpha, \beta \in A$ (2) and Proposition 2.4.1.2). Thus $\widehat{x}_\alpha = 0$ on $\sigma(F)\backslash A_\alpha$ (1), Proposition 4.1.2.21 a $\Rightarrow$ c, and Corollary 2.4.1.7 a)) and

$$A_\alpha \cap A_\beta = \emptyset$$

for all distinct $\alpha, \beta \in A$. From these relations and 1) we deduce that $\widehat{x} = \alpha$ on A_α for every $\alpha \in A$ and $\widehat{x} = 0$ on $\sigma(F)\backslash \bigcup\limits_{\alpha\in A} A_\alpha$. Hence

$$\sigma(x) = \widehat{x}(\sigma(F)) \subset A \cup \{0\}$$

(Corollary 2.4.1.7 a)). If 0 is an isolated point of $\sigma(x)$, then $\sigma(x)$ is finite and

$$\sum_{\alpha\in A} \widehat{x}_\alpha = \widehat{\sum_{\alpha\in A} x_\alpha} = e_{\sigma(F)}$$

by 3) (and Proposition 2.4.1.2). In this case,

$$\bigcup_{\alpha\in A} A_\alpha = \sigma(F)$$

and

$$0 \in \sigma(x) = \widehat{x}(\sigma(F)) \subset A$$

(Corollary 2.4.1.7 a)), which completes the proof of a).

We next prove c & d. If $0 \in A$, then $\sigma(x)$ is finite and

$$\sum_{\alpha\in A} \widehat{x}_\alpha = e_{\sigma(F)}$$

by 3), i.e.

$$\bigcup_{\alpha\in A} A_\alpha = \sigma(F)\,.$$

We deduce that

$$\widehat{f_\alpha(x)} = f_\alpha \circ \widehat{x} = \begin{cases} 1 & \text{on} \quad A_\alpha \\ 0 & \text{on} \quad \sigma(F)\backslash A_\alpha \end{cases}$$

whether or not $0 \in A$ (Theorem 4.1.3.1 c)), so that

$$\widehat{f_\alpha(x)} = \widehat{x}_\alpha\,, \quad f_\alpha(x) = x_\alpha$$

for every $\alpha \in A$ (Corollary 4.1.2.5). It follows that x and $\{x_\alpha \mid \alpha \in A\}$ generate the same closed subalgebra of E (Theorem 4.1.3.1 e)) and

$$F = E(x, 1)\,.$$

We now prove e). Take $\alpha \in A$. Then

$$ff_\alpha = f(\alpha)f_\alpha\,,$$

and so

$$f(\alpha)x_\alpha = f(\alpha)f_\alpha(x) = (f(\alpha)f_\alpha)(x) = (ff_\alpha)(x)\,.$$

We deduce that

$$f(x) - \sum_{\alpha\in B} f(\alpha)x_\alpha = f(x) - \sum_{\alpha\in B}(ff_\alpha)(x) = \left(f - \sum_{\alpha\in B} ff_\alpha\right)(x)$$

for any finite subset B of $\sigma(x)$. If $\sigma(x)$ is finite, then

$$f - \sum_{\alpha\in\sigma(x)} ff_\alpha = 0\,,$$

so that

$$f(x) = \sum_{\alpha\in\sigma(x)} f(\alpha)x_\alpha\,.$$

Now suppose that $\sigma(x)$ is infinite. Take $\varepsilon > 0$. Then $f(0) = 0$ and there is a finite subset B of $\sigma(x)$, such that

$$\sup_{\alpha\in\sigma(x)\backslash B} |f(\alpha)| < \varepsilon\,.$$

Thus

$$\left\| f(x) - \sum_{\alpha\in C} f(\alpha)x_\alpha \right\| = \left\| \left(f - \sum_{\alpha\in C} ff_\alpha\right)(x) \right\| = \left\| f - \sum_{\alpha\in C} ff_\alpha \right\| \leq \varepsilon$$

for any finite subsets C of $\sigma(x)$ containing B (Theorem 4.1.3.1). Since ε is arbitrary,

$$f(x) = \sum_{\alpha\in\sigma(x)} f(\alpha)x_\alpha\,. \qquad \blacksquare$$

Corollary 4.1.4.7 *Let E be a unital C^*-algebra. Take $x \in E$, $P \in \mathbb{K}[t]$, and put*

$$A := \{\alpha \in \mathbb{K} \mid P(\alpha) = 0\}.$$

Then the following are equivalent:

a) *x is self-normal and $P(x) = 0$.*

b) *There is a family $(x_\alpha)_{\alpha \in A}$ in $\Pr E$ such that*

1) $x = \sum_{\alpha \in A} \alpha x_\alpha$,

2) $\alpha, \beta \in A \Rightarrow x_\alpha x_\beta = \delta_{\alpha\beta} x_\alpha$,

3) $\sum_{\alpha \in A} x_\alpha = 1$.

If these conditions are fulfilled, then:

c) $\sigma(x) = \{\alpha \in A \mid x_\alpha \neq 0\}$.

d) *If $\alpha \in A$ and*

$$f_\alpha : \sigma(x) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \delta_{\alpha\beta}$$

then

$$f_\alpha(x) = x_\alpha.$$

In particular the family $(x_\alpha)_{\alpha \in A}$ in b) is unique.

e) *$E(x)$ is the C^*-subalgebra generated by $\{x_\alpha \mid \alpha \in A\}$. It is unital and finite-dimensional.*

f) $f \in C(\sigma(x)) \Rightarrow f(x) = \sum_{\alpha \in \sigma(x)} f(\alpha) x_\alpha$.

g) *If $A \subset \mathbb{R}$ (resp. $A \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}$), then x is selfadjoint (resp. unitary).*

a $\Rightarrow$ b & c & d & e & f & g. Since

$$\sigma(x) \subset A$$

(Theorem 2.1.3.4 a)), the assertions follow from Theorem 4.1.4.6 a $\Rightarrow$ b & c & d & e and Corollary 4.1.2.13 b $\Rightarrow$ a.

b $\Rightarrow$ a. By Theorem 4.1.4.6 b $\Rightarrow$ a & e, x is self-normal and

$$P(x) = \sum_{\alpha \in \sigma(x)} P(\alpha) x_\alpha = 0.$$

■

Corollary 4.1.4.8 *Let E be a unital complex C^*–algebra. Take $x \in E$ and $n \in \mathbb{N}$. Then the following are equivalent:*

a) *x is normal and $x^n = 1$.*

b) *There is a family $(x_j)_{j\in\mathbb{N}_n}$ in $\Pr E$ such that*

1) $x = \sum_{j=1}^{n} e^{\frac{2\pi ij}{n}} x_j$,

2) $j, k \in \mathbb{N}_n \Rightarrow x_j x_k = \delta_{jk} x_j$,

3) $\sum_{j=1}^{n} x_j = 1$.

If these equivalent conditions hold, then:

c) $\sigma(x) = \left\{ e^{\frac{2\pi ij}{n}} \mid j \in \mathbb{N}_n ,\ x_j \neq 0 \right\}$.

d) *If $j \in \mathbb{N}_n$ and*

$$f_j : \sigma(x) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \begin{cases} 1 & \text{if} \quad \alpha = e^{\frac{2\pi ij}{n}} \\ 0 & \text{if} \quad \alpha \neq e^{\frac{2\pi ij}{n}} \end{cases}$$

then

$$x_j = f_j(x) .$$

In particular, the family $(x_j)_{j\in\mathbb{N}_n}$ in b) is unique.

e) *$E(x)$ is the C^*–subalgebra of E generated by $\{x_j \mid j \in \mathbb{N}_n\}$; its dimension is at most n.*

f) $f \in \mathcal{C}(\sigma(x)) \Rightarrow \sum_{\alpha\in\sigma(x)} f(\alpha) x_\alpha = \sum_{\substack{j=1 \\ x_j \neq 0}}^{n} f\left(e^{\frac{2\pi ij}{n}}\right) x_j$.

g) *x is unitary.* ∎

Corollary 4.1.4.9 *Let E be a unital complex C^*–algebra. Take $x \in E$. Then the following are equivalent:*

a) *x is normal and $e^x = 1$.*

b) *There is a family $(x_n)_{n\in\mathbb{Z}}$ in $\Pr E$ such that*

1) $x = \sum_{n\in\mathbb{Z}} 2\pi i n x_n$,

2) $m, n \in \mathbb{Z} \Rightarrow x_m x_n = \delta_{mn} x_m$,

3) $\sum_{n\in\mathbb{Z}} x_n = 1$.

If these equivalent conditions apply, then:

c) $\sigma(x) = \{2\pi i n \mid n \in \mathbb{Z},\ x_n \neq 0\}$.

d) *If* $n \in \mathbb{Z}$ *and*

$$f_n : \sigma(x) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \delta_{\alpha,2\pi i n},$$

then

$$f_n(x) = x_n.$$

In particular, the family $(x_n)_{n\in\mathbb{Z}}$ *in b) is unique.*

e) $E(x)$ *is the* C^**-algebra generated by* $\{x_n \mid n \in \mathbb{Z}\}$. *It is unital and finite-dimensional.*

f) $f \in \mathcal{C}(\sigma(x)) \Rightarrow f(x) = \sum_{\substack{n\in\mathbb{Z}\\ x_n\neq 0}} f(2\pi i n)x_n$.

a $\Rightarrow$ b & c & d & e & f. By Corollary 4.1.3.16,

$$\sigma(x) \subset 2\pi i\mathbb{Z}$$

and the assertion follows from Theorem 4.1.4.6 a $\Rightarrow$ b & c & d & e and Corollary 2.2.4.5.

b $\Rightarrow$ a. By Theorem 4.1.4.6 b $\Rightarrow$ a, x is normal and

$$\sigma(x) \subset 2\pi i\mathbb{Z},$$

so that

$$e^x = 1$$

by Corollary 4.1.3.16. ■

Corollary 4.1.4.10 *Let* E *be a finite-dimensional unital complex (real)* C^**-algebra. Then the vector subspace of* E *generated by* $\operatorname{Pr} E$ *is* E *(is* $\operatorname{Re} E$*). In particular, if* A *is a linearly independent subset of* E *(of* $\operatorname{Re} E$*), then there is an algebraic basis* B *of the underlying vector space of* E *(of* $\operatorname{Re} E$*) with*

$$A \subset B, \quad B\backslash A \subset \operatorname{Pr} E.$$

Let $x \in \operatorname{Re} E$. By Corollary 2.1.3.5, $\sigma(x)$ is finite, so by Theorem 4.1.4.6 a $\Rightarrow$ b, x belongs to the vector subspace of E generated by $\operatorname{Pr} E$. Since x is arbitrary, the vector subspace of E generated by $\operatorname{Pr} E$ is E (is $\operatorname{Re} E$). The last assertion follows from the first one. ∎

Remark. Every finite-dimensional C^*-algebra possesses a unit (Corollary 4.2.8.8).

Example 4.1.4.11 *Take $n \in \mathbb{N}$ and let u be a self-normal element of the involutive algebra $\mathbb{K}_{n,n}$ defined in Example 2.3.1.32. Given $x, y \in \mathbb{K}^n$, define*

$$\langle x|y\rangle := \sum_{i=1}^{n} x_i \overline{y_i}\,,$$

$$[x|y] := [x_i \overline{y_j}]_{i,j \in \mathbb{N}_n}\,.$$

Then there are a finite subset A of $\mathbb{K}^n$ and a function $f : A \to \mathbb{K}$ such that

$$\langle x|y\rangle = \delta_{x,y}$$

for all $x, y \in A$ and

$$u = \sum_{x \in A} f(x)[x|x]\,, \quad \sigma(u) = f(A)\,.$$

If u is selfadjoint and

$$\widetilde{u} : \mathbb{K}_{n,n} \longrightarrow \mathbb{K}\,, \quad [\alpha_{i,j}]_{i,j \in \mathbb{N}_n} \longmapsto \sum_{i,j=1}^{n} \alpha_{ij} u_{ji}\,,$$

then

$$\|\widetilde{u}\| = \sum_{x \in A} |f(x)|\,.$$

We first discuss the complex case. Then as involutive algebra, $\mathbb{C}_{n,n}$ may be identified with the C^*-algebra $\mathcal{L}(\mathbb{C}^n)$ (Example 4.1.1.7). By Theorem 4.1.4.6 a $\Rightarrow$ b & c, there is a family $(u_\alpha)_{\alpha \in \sigma(u)}$ in $\operatorname{Pr} \mathcal{L}(\mathbb{C}^n) \backslash \{0\}$ such that

$$u = \sum_{\alpha \in \sigma(u)} \alpha u_\alpha\,,$$

$$\alpha, \beta \in \sigma(u) \Longrightarrow u_\alpha u_\beta = \delta_{\alpha\beta} u_\alpha\,.$$

By Example 2.3.1.36, given $\alpha \in \sigma(u)$, there is a nonempty finite subset A_α of $\mathbb{C}^n$ such that

$$\langle x|y\rangle = \delta_{x,y}$$

whenever $x, y \in A_\alpha$ and that

$$u_\alpha = \sum_{x \in A_\alpha} [x|x]\,.$$

Define

$$A := \bigcup_{\alpha \in \sigma(u)} A_\alpha\,,$$

$$f := \sum_{\alpha \in \sigma(u)} \alpha e^A_{A_\alpha}\,.$$

Take distinct $\alpha, \beta \in \sigma(u)$ and $x \in A_\alpha$, $y \in A_\beta$. Then

$$\langle x|y\rangle = \langle u_\alpha x|u_\beta y\rangle = \sum_{i=1}^n (u_\alpha x)_i \overline{(u_\beta y)}_i = \sum_{i=1}^n \left(\sum_{j=1}^n u_{\alpha ij} x_j\right)\left(\sum_{k=1}^n \overline{u}_{\beta ik}\overline{y}_k\right) =$$

$$= \sum_{i=1}^n \left(\sum_{j=1}^n u_{\alpha ij} x_j\right)\left(\sum_{k=1}^n u_{\beta ki}\overline{y}_k\right) = \sum_{j,k=1}^n x_j \overline{y}_k \left(\sum_{i=1}^n u_{\beta ki} u_{\alpha ij}\right) = 0\,.$$

Hence

$$\langle x|y\rangle = \delta_{x,y}$$

for all $x, y \in A$. Then

$$u = \sum_{\alpha \in \sigma(u)} \alpha \sum_{x \in A_\alpha} [x|x] = \sum_{x \in A} f(x)[x|x]\,,$$

$$f(A) = \{\alpha \mid \alpha \in \sigma(u)\} = \sigma(u)\,.$$

We now turn to the real case. The matrix u may be viewed as an element of $\mathbb{C}_{n,n}$. Since in this case u is selfadjoint by assumption, we have that $\sigma(u) \subset \mathbb{R}$ and the assertion follows from the considerations above.

We now prove the final assertion. Take $v \in \operatorname{Re} \mathbb{K}^\#_{n,n}$. By the considerations above, there is a finite subset B of $\mathbb{K}^n$ and a function $g : B \to [-1, 1]$ such that for every $x, y \in B$,

$$\langle x|y\rangle = \delta_{x,y}$$

and

$$v = \sum_{y\in B} g(y)[y|y]\,.$$

Then

$$|\widetilde{u}(v)| = \left|\sum_{x\in A} f(x)\left(\sum_{y\in B} g(y)|\langle x|y\rangle|^2\right)\right| \leq$$

$$\leq \sum_{x\in A} |f(x)|\left(\sum_{y\in B} |\langle x|y\rangle|^2\right) \leq \sum_{x\in A} |f(x)|\,.$$

Since v is arbitrary, it follows by Proposition 2.3.2.22 j),

$$\|\widetilde{u}\| \leq \sum_{x\in A} |f(x)|\,.$$

Defining

$$h : A \longrightarrow \mathbb{K}\,, \quad x \longmapsto \begin{cases} 1\,, & \text{if } f(x) \geq 0 \\ -1\,, & \text{if } f(x) < 0\,, \end{cases}$$

$$w := \sum_{y\in A} h(y)[y|y]\,,$$

the above considerations show that $\|w\| \leq 1$. Thus

$$\|\widetilde{u}\| \geq \widetilde{u}(w) = \sum_{x\in A} f(x)\left(\sum_{y\in A} h(y)|\langle x|y\rangle|^2\right) = \sum_{x\in A} f(x)h(x) = \sum |f(x)|\,.$$

Hence

$$\|\widetilde{u}\| = \sum_{x\in A} |f(x)|\,. \qquad \blacksquare$$

Proposition 4.1.4.12 *Let p be an orthogonal projection of the C^*-algebra E. If p is not a unit of E, then there is an $x \in \operatorname{Re} E$ for which*

$$x \neq 0\,, \quad px = xp = 0\,.$$

Since p is not a unit of E, there is some $y \in E \backslash pEp$. Put

$$y_1 := (1-p)y^*y(1-p)\,, \quad y_2 := (1-p)yy^*(1-p)\,.$$

Then $y_1, y_2 \in \operatorname{Re} E$ and

$$py_1 = y_1p = py_2 = y_2p = 0\,.$$

If $y_1 = y_2 = 0$, then

$$\|y(1-p)\|^2 = \|(1-p)y^*y(1-p)\| = \|y_1\| = 0\,,$$

$$\|(1-p)y\|^2 = \|(1-p)yy^*(1-p)\| = \|y_2\| = 0\,,$$

so that

$$y = yp = py\,,$$

$$y = pyp \in pEp\,,$$

which is a contradiction. Hence either $y_1 \neq 0$ or $y_2 \neq 0$. We put $x := y_1$ in the first case and $x := y_2$ in the second case. ■

Corollary 4.1.4.13 *Let F be a Gelfand C^*-subalgebra of the C^*-algebra E. If $\sigma(F)$ is compact and F does not contain a unit of E, then there is a Gelfand C^*-subalgebra G of E containing F and an $x' \in \sigma(G)$ which is identically 0 on F.*

Using the Gelfand transform on F, we find an orthogonal projection p of F with

$$px = xp = x$$

for every $x \in F$. Since p is not a unit of E, there is an $x \in \operatorname{Re} E$, with

$$x \neq 0\,, \quad px = xp = 0$$

(Proposition 4.1.4.12). By Corollary 4.1.2.3, the C^*-subalgebra G of E generated by $F \cup \{x\}$ is a Gelfand C^*-algebra. Using the Gelfand transform on G, we see that there is an $x' \in \sigma(G)$ such that $x'(x) \neq 0$. It follows $x'(p) = 0$, i.e. x' is identically 0 on F. ■

Corollary 4.1.4.14 *Let E be a C^*–algebra and $x \in \operatorname{Sn} E$ such that $E(x)$ does not contain a unit of E. Then there is a Gelfand C^*–subalgebra F of E containing x such that*

$$\inf_{x' \in \sigma(F)} |x'(x)| = 0 .$$

If $\sigma(E(x))$ is not compact, then we may take $F := E(x)$ since $\overset{F}{\widehat{x}} \in C_0(\sigma(F))$ (Theorem 2.4.1.3 e)). If $\sigma(E(x))$ is compact, then by Corollary 4.1.4.13, there is a Gelfand C^*–subalgebra F of E containing $E(x)$ and an $x' \in \sigma(G)$ which is identically 0 on $E(x)$. In particular, $x \in F$ and $x'(x) = 0$. ■

4.2 The Order Relation

Every C^*–algebra is equipped with a canonical ordering with good properties, albeit lacking a lattice structure. Together with the functional calculus, this ordering plays a central role in the theory of C^*–algebras. For example every positive element has a uniqueness square root, and so we can define the absolute value of any arbitrary element. Another example is the existence of a canonical approximate unit, which is the upper section filter of an upward directed set. Finally the ordering is also used in the Segal–Schatten Theorem to show that any quotient algebra of a C^*–algebra is again a C^*–algebra. A consequence of this highlights are the strong properties of C^*–algebras: If $u : E \to F$ is a homomorphism of involutive algebras between the C^*–algebras E and F, then $u(E)$ is a C^*–subalgebra of F and the associated map $E/\operatorname{Ker} u \to u(E)$ is an isometry of C^*–algebras.

In this section, E is always a C^*–algebra.

4.2.1 Definition and General Properties

Theorem 4.2.1.1 (0) (Fukamiya 1952, Kelley–Vaught 1953) *There is a unique order relation on E which renders it an ordered Banach space with*

$$E_+ = \{x \in \operatorname{Re} E \mid \sigma(x) \subset \mathbb{R}_+\} \ (= \{x \in \operatorname{No} E \mid \sigma(x) \subset \mathbb{R}_+\} \text{ if } \mathbb{K} = \mathbb{C})$$

By Theorem 4.1.1.13, we may assume that E is unital. Since E is symmetric (Theorem 4.1.2.1) and since

$$\|x\| = r(x)$$

for every $x \in \operatorname{Re} E$ (Theorem 4.1.1.16 b)), the assertion follows from Proposition 2.3.2.34 b) (and Corollary 4.1.2.13 b $\Rightarrow$ a). ■

Definition 4.2.1.2 (0) *The order relation on E introduced in Theorem 4.2.1.1 is called* ***the canonical order relation of*** *E.*

In the sequel, all C^*-algebras are taken to be endowed with their canonical orders.

The order relation of E remains unchanged if we reverse the multiplication.

The example $E := c_0$ shows that it may happen that E is order complete but its associated unital C^*–algebra is not.

Corollary 4.2.1.3 *If E is unital and $x \in E_+$, then the following are equivalent:*

a) *x is not invertible.*

b) *There is an $x' \in \tau(E)$ with $x'(x) = 0$.*

c) *There is an $x' \in \tau_0(E)$ with $x'(x) = 0$.*

c $\Rightarrow$ b is trivial.
b $\Rightarrow$ a $\Rightarrow$ c follows from Corollary 4.1.2.7 b). ■

Corollary 4.2.1.4 (0) *Let E, F be C^*-algebras and $u : E \to F$ a homomorphism of involutive algebras. Then*

$$u(E_+) \subset F_+ .$$

In particular,

$$x, y \in E\,,\ x \leq y \Longrightarrow ux \leq uy\,.$$

Given $x \in E$,

$$\sigma_F(ux) \subset \sigma_E(x) \cup \{0\}$$

(Corollary 2.1.3.12) and the statement follows from the fact that $u(\mathrm{Re}\, E) \subset \mathrm{Re}\, F$ (Proposition 2.3.1.7). The last assertion follows from the first one. ■

Proposition 4.2.1.5 *If E is not unital and $\mathbb{K} \times E$ is its associated unital C^*-algebra, then*

$$\mathbb{R}_+ \times E_+ \subset (\mathbb{K} \times E)_+ \subset \mathbb{R}_+ \times E\,, \quad \{0\} \times E_+ = (\{0\} \times E) \cap (\mathbb{K} \times E)_+\,.$$

In particular, if A is a subset of E, x its infimum in E, and (α, y) the infimum of $\{0\} \times A$ in $\mathbb{K} \times E$, then $\alpha = 0$ and $x = y$.

If $(\alpha, x) \in \mathbb{R}_+ \times E_+$, then

$$(\alpha, x) = (\alpha, 0) + (0, x) \in (\mathbb{K} \times E)_+\,,$$

so that

$$\mathbb{R}_+ \times E_+ \subset (\mathbb{K} \times E)_+\,.$$

Now take $(\alpha, x) \in (\mathbb{K} \times E)_+$. Since $0 \in \sigma((0, x))$,

$$\alpha \in \alpha + \sigma((0,x)) = \sigma((\alpha,x)) \subset \mathbb{R}_+ .$$

Hence

$$(\mathbb{K} \times E)_+ \subset \mathbb{R}_+ \times E .$$

The relation

$$\{0\} \times E_+ = (\{0\} \times E) \cap (\mathbb{K} \times E)_+$$

follows from

$$x \in E \Longrightarrow \sigma_{\mathbb{K}\times E}((0,x)) = \sigma_E(x) .$$

We now prove the last assertion. Take $z \in A$. Then

$$(\alpha, y) \leq (0, z) ,$$

$$(-\alpha, z - y) \in (\mathbb{K} \times E)_+ \subset \mathbb{R}_+ \times E ,$$

$$\alpha \leq 0 .$$

Since $(0,x)$ is a lower bound for $\{0\} \times A$,

$$(0,x) \leq (\alpha, y) ,$$

$$(\alpha, y - x) \in (\mathbb{K} \times E)_+ \subset \mathbb{R}_+ \times E ,$$

$$\alpha \geq 0 ,$$

$$\alpha = 0 ,$$

$$x = y . \qquad \blacksquare$$

Remark. Take $E := c_0$ and $A := \{-e_n \mid n \in \mathbb{N}\}$. Then $(-1,0)$ is the infimum of $\{0\} \times A$ in $\mathbb{K} \times E$ and A has no infimum in E.

Proposition 4.2.1.6 (0) $x \in E_+ \Rightarrow \|x\| \in \sigma(x) \subset [0, \|x\|]$.

This statement follows immediately from

$$||x|| = r(x)$$

(Theorem 4.1.1.16 b)). $\blacksquare$

Corollary 4.2.1.7 (0) *If f is a continuous increasing positive real function on $\mathbb{R}_+$, then*

$$\|f(x)\| = f(\|x\|)$$

for every $x \in E_+$.

We have

$$\|f(x)\| = \|f|\sigma(x)\| = f(\|x\|)$$

(Theorem 4.1.3.1 b), Proposition 4.2.1.6). ■

Proposition 4.2.1.8 (0) *If F is a C^*-subalgebra of E then*

$$F_+ = E_+ \cap F.$$

In particular,

$$x \leq y \text{ in } E \Longleftrightarrow x \leq y \text{ in } F$$

for all $x, y \in F$.

The first claim follows from

$$x \in F \Longrightarrow \sigma_E(x) = \sigma_F(x)$$

(Theorem 4.1.2.12). The second assertion follows from the first one. ■

Proposition 4.2.1.9 (0) *If F is a Gelfand C^*-subalgebra of E, then*

$$x \leq y \Longleftrightarrow \overset{F}{\widehat{x}} \leq \overset{F}{\widehat{y}}$$

for all $x, y \in F$.

We have

$$\overset{F}{\widehat{x}}\,(\sigma(F)) \subset \sigma_F(x) \subset \overset{F}{\widehat{x}}(\sigma(F)) \cup \{0\}$$

(Corollary 2.4.1.7 a)), so that

$$x \in F_+ \Longleftrightarrow \overset{F}{\widehat{x}} \geq 0$$

for all $x \in F$. The proposition now follows. ■

Corollary 4.2.1.10 *If* $\{x_1, x_2, y_1, y_2\}$ *is a commutative set of* E_+ *with*

$$x_1 \leq y_1\,, \quad x_2 \leq y_2\,,$$

then

$$x_1 x_2 \leq y_1 y_2\,.$$

Let F be the C^*-subalgebra of E generated by $\{x_1, x_2, y_1, y_2\}$. It is a Gelfand C^*-algebra (Corollary 4.1.2.3). By Proposition 4.2.1.9,

$$0 \leq \overset{F}{\widehat{x}}_1 \leq \overset{F}{\widehat{y}}_1, \quad 0 \leq \overset{F}{\widehat{x}}_2 \leq \overset{F}{\widehat{y}}_2\,,$$

so that

$$0 \leq \overset{F}{\widehat{x}}_1 \overset{F}{\widehat{x}}_2 \leq \overset{F}{\widehat{y}}_1 \overset{F}{\widehat{y}}_2\,,$$

$$\overset{F}{\widehat{x_1 x_2}} \leq \overset{F}{\widehat{y_1 y_2}}\,,$$

$$x_1 x_2 \leq y_1 y_2\,,$$

by Proposition 4.1.2.9. ■

Corollary 4.2.1.11 (0) *Let* A *be a subset of* $\mathbb{R}$, f *an increasing continuous real function on* A. *Take* $x, y \in \operatorname{Re} E$ *such that*

$$\sigma(x) \subset A\,, \quad \sigma(y) \subset A\,, \quad x \leq y\,, \quad xy = yx\,.$$

Then

$$f(x) \leq f(y)\,.$$

Let F be the C^*-subalgebra generated by $\{x, y\}$. It is a Gelfand C^*-algebra (Corollary 4.1.2.3). By Proposition 4.2.1.9,

$$\overset{F}{\widehat{x}} \leq \overset{F}{\widehat{y}}$$

so that

$$\overset{F}{\widehat{f(x)}} = f \circ \overset{F}{\widehat{x}} \leq f \circ \overset{F}{\widehat{y}} = \overset{F}{\widehat{f(y)}}$$

(Corollary 4.1.3.5) and

$$f(x) \leq f(y)$$

(Proposition 4.2.1.9). ■

Corollary 4.2.1.12 (0) *If* $x \in \operatorname{Sn} E$ *and* $f, g \in \mathcal{C}(\sigma(x))$, *then*

$$f \leq g \Longleftrightarrow f(x) \leq g(x)\,.$$

We may assume that E is unital. Put $F := E(x,1)$. Then F is a Gelfand C^*–algebra (Corollary 4.1.2.3) and the map

$$\sigma(F) \longrightarrow \sigma(x)\,, \quad x' \longmapsto \overset{F}{\widehat{x}}(x')$$

is a homeomorphism (Corollary 4.1.2.4). Hence

$$f \leq g \Longleftrightarrow \overset{F}{\widehat{f(x)}} = f \circ \overset{F}{\widehat{x}} \leq g \circ \overset{F}{\widehat{x}} = \overset{F}{\widehat{g(y)}} \Longleftrightarrow f(x) \leq g(x)$$

by Corollary 4.1.3.5 and Proposition 4.2.1.9. ■

Corollary 4.2.1.13 (0) *Let* E *be unital and take* $x \in E$ *with* $x \geq 1$. *Then* x *is invertible and*

$$0 \leq x^{-1} \leq 1\,.$$

We have

$$\sigma(x) = \sigma(x-1) + 1 \subset [1, \infty[\,.$$

Hence x is invertible. By Corollary 4.2.1.12,

$$0 \leq x^{-1} \leq 1\,.$$ ■

Remark. This result will be generalized in Proposition 4.2.4.6.

Proposition 4.2.1.14 *Take* $x \in E_+$ *with* $\|x\| = 1$. *If*

$$y \in E_+\,,\ y \leq x \Longrightarrow \exists\, \alpha \in \mathbb{R}_+\,,\ y = \alpha x\,,$$

then x *is idempotent.*

We have

$$1 \in \sigma(x) \subset [0,1]$$

(Proposition 4.2.1.6). Define

$$f : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \alpha\ ,$$

$$g : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \alpha^2\,.$$

Then $0 \leq g \leq f$, so that

$$0 \leq g(x) \leq f(x)$$

(Corollary 4.2.1.12). Since

$$g(x) = x^2\,, \quad f(x) = x\,,$$

(Theorem 4.1.3.1 a)), there is an $\alpha \in \mathbb{R}_+$, with

$$x^2 = \alpha x\,.$$

We deduce that

$$\alpha = \|\alpha x\| = \|x^2\| = \|g(x)\| = g(\|x\|) = 1$$

(Theorem 4.1.3.1 b), Corollary 4.2.1.7), i.e. x is idempotent. ■

Proposition 4.2.1.15 (0) *Given* $x \in \operatorname{Re} E$,

$$\sup_{\alpha\in\sigma(x)} \alpha = \inf\{\alpha \in \mathbb{R} \mid x \leq \alpha 1\}\,,$$

$$\inf_{\alpha\in\sigma(x)} \alpha = \sup\{\alpha \in \mathbb{R} \mid \alpha 1 \leq x\}\,.$$

Take $\beta \in \mathbb{R}$. Then

$$\sigma(\beta 1 - x) = \beta - \sigma(x)\,, \quad \sigma(x - \beta 1) = \sigma(x) - \beta\,,$$

so that

$$x \leq \beta 1 \Longleftrightarrow \beta 1 - x \geq 0 \Longleftrightarrow \beta - \sigma(x) \subset \mathbb{R}_+ \Longleftrightarrow \sup_{\alpha\in\sigma(x)} \alpha \leq \beta\,,$$

$$\beta 1 \leq x \Longleftrightarrow x - \beta 1 \geq 0 \Longleftrightarrow \sigma(x) - \beta \subset \mathbb{R}_+ \Longleftrightarrow \inf_{\alpha\in\sigma(x)} \alpha \geq \beta\,.$$ ■

Corollary 4.2.1.16 (0) *Take* $x \in \operatorname{Re} E$ *and* $\alpha \in \mathbb{R}_+$. *Then the following are equivalent:*

a) $\|x\| \leq \alpha$.

b) $-\alpha 1 \leq x \leq \alpha 1$.

c) $\sigma(x) \subset [-\alpha, \alpha]$.

a $\Leftrightarrow$ b. $F := E(x,1)$ is a Gelfand C^*–algebra (Corollary 4.1.2.3). Thus

$$-\alpha 1 \leq x \leq \alpha 1 \Longleftrightarrow -\alpha e_{\sigma(F)} \overset{F}{\leq} \widehat{x} \leq \alpha e_{\sigma(F)} \Longleftrightarrow \|\widehat{x}\| \leq \alpha \Longleftrightarrow \|x\| \leq \alpha$$

(Proposition 4.2.1.9, Corollary 4.1.2.5).

b $\Leftrightarrow$ c follows from Proposition 4.2.1.15. ■

Corollary 4.2.1.17 (0) *Given $x \in E_+$, then following are equivalent:*

a) $x \in E_+^{\#}$.

b) $x \leq 1$.

c) $1 - x \in E_+^{\#}$.

a $\Leftrightarrow$ b follows from Corollary 4.2.1.16 a $\Leftrightarrow$ b.

b $\Rightarrow$ c. Since

$$0 \leq 1 - x \leq 1,$$

c) follows from b $\Rightarrow$ a.

c $\Rightarrow$ b is trivial. ■

Corollary 4.2.1.18 (0) *Take $x, y \in \operatorname{Re} E$. If*

$$-y \leq x \leq y,$$

then

$$\|x\| \leq \|y\|.$$

By Corollary 4.2.1.16 a $\Leftrightarrow$ b,

$$-\|y\|1 \leq -y \leq x \leq y \leq \|y\|1,$$

$$\|x\| \leq \|y\|. \quad ■$$

Corollary 4.2.1.19 (0) *Let $(x_\iota)_{\iota \in I}$, $(y_\iota)_{\iota \in I}$ be families in E such that $(y_\iota)_{\iota \in I}$ is summable and*

$$-y_\iota \leq x_\iota \leq y_\iota$$

for every $\iota \in I$. Then $(x_\iota)_{\iota \in I}$ is also summable and

$$-\sum_{\iota \in I} y_\iota \leq \sum_{\iota \in I} x_\iota \leq \sum_{\iota \in I} y_\iota,$$

$$\|\sum_{\iota \in I} x_\iota\| \leq \|\sum_{\iota \in I} y_\iota\|.$$

The assertion follows immediately from Corollary 4.2.1.18 and Corollary 1.7.1.6 b). ■

Corollary 4.2.1.20 (0) *If E is a finite-dimensional C^*-algebra then there is an $\alpha \in \mathbb{R}_+$ such that*

$$\sum_{\iota \in I} \|x_\iota\|^2 \leq \alpha \left\| \sum_{\iota \in I} x_\iota^* x_\iota \right\|$$

for every family $(x_\iota)_{\iota \in I}$ in E for which $(x_\iota^* x_\iota)_{\iota \in I}$ is summable.

The assertion follows from the Corollaries 4.2.1.18 and 1.7.1.7. ■

Corollary 4.2.1.21 *If E is unital and x is the supremum in E of a subset A of $\operatorname{Re} E$, then*

$$\sup \sigma(x) = \sup_{y \in A} \sup \sigma(y), \quad \|x\| \leq \sup_{y \in A} \|y\|.$$

Define

$$\alpha := \sup_{y \in A} \sup \sigma(y).$$

By Proposition 4.2.1.15, $\alpha 1$ is an upper bound for A. Hence $x \leq \alpha 1$. Again by Proposition 4.2.1.15,

$$\sup \sigma(y) \leq \sup \sigma(x) \leq \alpha,$$

for every $y \in A$, so that

$$\alpha \leq \sup \sigma(x),$$

$$\alpha = \sup \sigma(x).$$

We set

$$\beta := \inf \sigma(x).$$

By Proposition 4.2.1.15,

$$\inf \sigma(y) \leq \beta$$

for every $y \in A$. Hence

$$\|x\| = r(x) = \sup\{|\alpha|, |\beta|\} \leq \sup_{y \in A} \sup\{|\sup \sigma(y)|, |\inf \sigma(y)|\} = \sup_{y \in A} \|y\|. \quad ■$$

Proposition 4.2.1.22 (0) $\Pr E \subset E_+$.

The assertion follows from Proposition 4.1.2.21 a $\Rightarrow$ c. ■

4.2.2 More about the Order Relation

Theorem 4.2.2.1 (0) *Given $x \in E$, the following are equivalent:*

a) $x \in E_+$.

b) *There is a* $y \in E_+$ *with* $x = y^2$.

c) *There is a* $y \in E$ *with* $x = y^*y$.

d) $x'(x) \in \mathbb{R}_+$ *whenever* x' *is a continuous positive linear form on* E.

a $\Rightarrow$b. Put

$$f : \sigma(x) \longrightarrow \mathbb{R}_+, \quad \alpha \longmapsto \sqrt{\alpha},$$

$$y := f(y).$$

Then $y \in E_+$ and

$$y^2 = f^2(x) = x$$

(Theorem 4.1.3.1 a),b), Corollary 4.2.1.12).

b $\Rightarrow$ c $\Rightarrow$ d is trivial.

d $\Rightarrow$ a. Let $\widetilde{E}$ be the unital C^*-algebra associated to E (Theorem 4.1.1.13). Then $x'(x) \in \mathbb{R}_+$ for every $x' \in \widetilde{E}'_+$. Thus

$$\sigma_E(x) = \sigma_{\widetilde{E}}(x) \subset \overset{\vee}{x}(\tau(\widetilde{E})) \subset \mathbb{R}_+$$

by Corollary 4.1.2.7 a). ■

Remark. In the complex case, the above theorem can be deduced also from Proposition 2.4.2.9 c).

Corollary 4.2.2.2 (0) *Let* $(x_\iota)_{\iota \in I}$ *be a family in* E_+. *If*

$$\inf_{\iota \in I} \|x_\iota\| = 0,$$

then

$$\bigwedge_{\iota \in I} x_\iota = 0.$$

In particular,

$$\bigwedge_{n \in \mathbb{N}} (\frac{1}{n} x) = 0$$

for every $x \in E_+$.

Let y be a lower bound for $(x_\iota)_{\iota\in I}$. Let x' be a continuous positive linear form on E. Then

$$x'(y) \leq x'(x_\iota) \leq \|x'\|\,\|x_\iota\|$$

for every $\iota \in I$ (Theorem 4.2.2.1 a $\Rightarrow$ d) and so

$$x'(y) \leq 0\,,$$

$$x'(-y) \in \mathbb{R}_+\,.$$

By Theorem 4.2.2.1 d $\Rightarrow$ a,

$$-y \in E_+\,,$$

$$y \leq 0\,.$$

Hence

$$\bigwedge_{\iota\in I} x_\iota = 0\,. \qquad \blacksquare$$

Corollary 4.2.2.3 (0) *If $x \in E_+$, then $y^*xy \in E_+$ for every $y \in E$. In particular,*

$$x, y, z \in E\,,\ x \leq y \Longrightarrow z^*xz \leq z^*yz\,,\ -\|x\|z^*z \leq z^*xz \leq \|x\|z^*z\,.$$

By Theorem 4.2.2.1 a $\Rightarrow$ b, there is a $z \in E_+$ with $z^2 = x$. Thus, by Theorem 4.2.2.1 c $\Rightarrow$ a,

$$y^*xy = y^*z^2y = (zy)^*(zy) \in E_+\,.$$

The final relation follows from the preceding one and Corollary 4.2.1.16 a $\Rightarrow$ b. $\blacksquare$

Corollary 4.2.2.4 (0) *Taking $x, y \in E$ and $z \in E_+$,*

$$x^*y + y^*x \leq x^*x + y^*y\,,$$

$$x^*zy + y^*zx \leq x^*zx + y^*zy\,.$$

We have

$$0 \leq (x-y)^*(x-y) = x^*x - x^*y - y^*x + y^*y\,,$$

$$0 \leq (x-y)^*z(x-y) = x^*zx - x^*zy - y^*zx + y^*zy$$

(Corollary 4.2.2.3). Hence

$$x^*y + y^*x \leq x^*x + y^*y\,,$$

$$x^*zy + y^*zx \leq x^*zx + y^*zy\,. \qquad \blacksquare$$

Corollary 4.2.2.5 (0) *If $x \in E_+^\#$ then for every $y \in E^\#$, $((y^*xy)^n)_{n\in\mathbb{N}}$ is a decreasing sequence in $E_+^\#$.*

By Corollary 4.2.2.3, $y^*xy \in E_+$ and from

$$\|y^*xy\| \leq \|y^*\|\,\|x\|\,\|y\| \leq 1$$

it follows that $y^*xy \in E_+^\#$. Since

$$r(y^*xy) = \|y^*xy\| \leq 1$$

(Theorem 4.1.1.16 b)), the claim follows from Corollary 4.2.1.12. $\blacksquare$

Corollary 4.2.2.6 (0) *If x' is a continuous positive linear from on E, then*

$$|x'(xy)|^2 \leq \|x\|\,\|y\|^2\|x'\|x'(x)$$

for every $x \in E_+$ and $y \in E$.

By Theorem 4.2.2.1 a $\Rightarrow$ b, there is a $z \in E_+$ with $x = z^2$. Thus

$$|x'(xy)|^2 \leq x'(zzy)^2 \leq x'(zz^*)x'((zy)^*(zy)) =$$

$$= x'(x)x'(y^*xy) \leq x'(x)\|x\|x'(y^*y) \leq$$

$$\leq x'(x)\|x\|\,\|x'\|\,\|y^*y\| = \|x\|\,\|y\|^2\|x'\|x'(x)$$

(Proposition 2.3.4.6 c), Corollary 2.3.4.8). $\blacksquare$

Corollary 4.2.2.7 (0) *If $x, y \in E$ and $\theta \in \mathbb{R}$, then*

$$\|(\cos\theta)x + (\sin\theta)y\|^2 \leq \|x^*x + y^*y\|\,.$$

By Theorem 4.2.2.1 c $\Rightarrow$ a,

$$0 \leq ((\cos\theta)x + (\sin\theta)y)^*((\cos\theta)x + (\sin\theta)y) \leq$$

$$\leq ((\cos\theta)x + (\sin\theta)y)^*((\cos\theta)x + (\sin\theta)y) +$$

$$+((\sin\theta)x - (\cos\theta)y)^*((\sin\theta)x - (\cos\theta)y) =$$

$$= (\cos^2\theta)x^*x + (\sin^2\theta)y^*y + \cos\theta\,\sin\theta(x^*y + y^*x) +$$

$$+(\sin^2\theta)x^*x + (\cos^2\theta)y^*y - \cos\theta\sin\theta((x^*y + y^*x) = x^*x + y^*y\,.$$

By Corollary 4.2.1.18,

$$\|(\cos\theta)x + (\sin\theta)y\|^2 = \|((\cos\theta)x + (\sin\theta)y)^*((\cos\theta)x + (\sin\theta)y)\| \leq$$

$$\leq \|x^*x + y^*y\|\,. \qquad \blacksquare$$

Proposition 4.2.2.8 *Let x' be a positive linear form on E and A a non-empty downward directed set of E_+.*

a) $\{x \in E \mid \inf_{y\in A} x'(x^*yx) = 0\}$ *is a vector subspace of E.*

b) *If $\mathbb{K} = \mathbb{C}$ and if*

$$\inf_{y\in A} x'(x^*yx) = 0$$

for all unitary elements x of E, then

$$\inf_{y\in A} x'(x^*yx) = 0$$

for all $x \in E$.

a) Let $x, y \in E$ such that

$$\inf_{z\in A} x'(x^*zx) = \inf_{z\in A} x'(y^*zy) = 0\,.$$

We have

$$(x+y)^*z(x+y) = x^*zx + y^*zy + x^*zy + y^*zx \leq 2(x^*zx + y^*zy)$$

for all $z \in E_+$ (Corollary 4.2.2.4). Hence

$$\inf_{z\in A} x'((x+y)^*z(x+y)) = 0\,.$$

b) follows from a) and Corollary 4.1.3.7. $\blacksquare$

Theorem 4.2.2.9 (0) *Take* $x \in \operatorname{Re} E$.

a) *There are uniquely determined* $y, z \in E_+$ *such that*

$$x = y - z\,, \quad yz = zy = 0\,.$$

Define

$$x^+ := y\,, \quad x^- := z$$

and call x^+ *and* x^- ***the positive*** *and* ***the negative part of*** x, *respectively.*

b)

$$x^+ = f(x)\,, \quad x^- = g(x)\,, \quad \|x\| = \sup\{\|x^+\|, \|x^-\|\}\,,$$

where

$$f : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \sup\{\alpha, 0\}\,,$$

$$g : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \sup\{-\alpha, 0\}\,.$$

c) x *and* (x^+, x^-) *generate the same closed real subalgebra of* E *and* $E(x)$ *is the* C^**-subalgebra of* E *generated by* $\{x^+, x^-\}$.

d) $(-x)^+ = x^-\,, \quad (-x)^- = x^+\,.$

e) *Let* F *be an* E*-module and* $a \in F$ *such that* $ax = 0$ *(resp.* $xa = 0$*). then*

$$ax^+ = ax^- = 0 \qquad (\textit{resp. } x^+a = x^-a = 0)\,.$$

a) and b).

Step 1 b) and the "existence" part in a)

Since $\sigma(x) \subset \mathbb{R}$ (Corollary 4.1.2.13 a $\Rightarrow$ b), f and g are well–defined. Put

$$y := f(x)\,, \quad z := g(x)\,.$$

Then $y, z \in E_+$ (Corollary 4.2.1.12) and

$$y - z = f(x) - g(x) = (f - g)(x) = x\,,$$

$$yz = f(x)g(x) = (fg)(x) = 0\,,$$

$$zy = 0\,,$$

$$\sup\{\|y\|, \|z\|\} = \sup\{\|f\|, \|g\|\} = r(x) = \|x\|$$

(Theorems 4.1.3.1 and 4.1.1.16 b)).

Step 2 the "uniqueness" part in a)

Take $y, z \in E_+$ with

$$x = y - z\,, \quad yz = zy = 0$$

and let F be the C^*-subalgebra of E generated by $\{y, z\}$. F is a Gelfand C^*-algebra (Corollary 4.1.2.3) and

$$\overset{F}{\widehat{x}} = \overset{F}{\widehat{y}} - \overset{F}{\widehat{z}}\,, \quad \overset{F}{\widehat{y}}\,\overset{F}{\widehat{z}} = 0\,.$$

Thus

$$\overset{F}{\widehat{f(x)}} = f \circ \overset{F}{\widehat{x}} = \overset{F}{\widehat{y}}\,, \quad \overset{F}{\widehat{g(x)}} = g \circ \overset{F}{\widehat{x}} = \widehat{z}$$

(Corollary 4.1.3.5). Hence

$$y = f(x)\,, \quad z = g(x)\,.$$

c) follows from b) and Corollary 4.1.3.3 a),

d) follows from a).

e) follows from b) and Theorem 4.1.3.1 f). ■

Corollary 4.2.2.10 (0) *A linear form x' on E is positive iff $x'(E_+) \subset \mathbb{R}_+$ (and $x' \in \operatorname{Re} E'$ if $\mathbb{K} = \mathbb{R}$).*

If x' is positive, then $x'(E_+) \subset \mathbb{R}_+$ by Theorem 4.2.2.1 a $\Rightarrow$ d. For the converse suppose that $x'(E_+) \subset \mathbb{R}_+$ (and $x' \in \operatorname{Re} E'$ if $\mathbb{K} = \mathbb{R}$). Then

$$x'(x^*x) \in \mathbb{R}_+$$

for every $x \in E$ by Theorem 4.2.2.1 c $\Rightarrow$ a. By Theorem 4.2.2.9 a),

$$x'(x) = x'(x^+) - x'(x^-) \in \mathbb{R}\,,$$

for every $x \in \operatorname{Re} E$. It follows that

$$x'(x^*) = x'(\operatorname{re} x - i \operatorname{im} x) = x' \operatorname{re} x) - ix'(\operatorname{im} x) = \overline{x'(\operatorname{re} x) + ix'(\operatorname{im} y)} = \overline{x'(x)}$$

for every $x \in E$. Hence x' is involutive and therefore positive. ■

Corollary 4.2.2.11 (0) *If $u : E \to F$ is a homomorphism of C^*-algebras then $u'(F'_+) \subset E'_+$.*

Take $y' \in F'_+$ and $x \in E_+$. By Corollary 4.2.1.4, $ux_+ \in F_+$, so that by Corollary 4.2.2.10,

$$\langle x, u'y' \rangle = \langle ux, y' \rangle \in \mathbb{R}_+ ,$$

$$u'y' \in E'_+ . \qquad \blacksquare$$

Corollary 4.2.2.12 *The maps*

$$\mathrm{Re}\, E \longrightarrow E_+ , \quad x \longmapsto x^+ ,$$

$$\mathrm{Re}\, E \longrightarrow E_+ , \quad x \longmapsto x^-$$

are continuous.

The corollary follows immediately from Theorem 4.2.2.9 b) and Proposition 4.1.3.20. $\blacksquare$

Corollary 4.2.2.13 (0) *Every positive linear form x' on E is continuous and*

$$\|x'\| = \sup_{x \in E_+^\#} x'(x) .$$

By Theorem 4.2.2.9 b), the absolute convex hull of $E_+^\#$ is a 0–neighbourhood in $\mathrm{Re}\, E$. Thus by Corollary 1.7.1.8 b), $x'|\mathrm{Re}\, E$ is continuous. If $\mathbb{K} = \mathbb{C}$, then by Proposition 2.3.2.7, x' is continuous. Assume $\mathbb{K} = \mathbb{R}$ and let $x \in E^\#$. Then $\frac{1}{2}(x + x^*) \in \mathrm{Re}\, E^\#$ and

$$x'(\frac{1}{2}(x - x^*)) = x'^*(\frac{1}{2}(x - x^*)) = x'(\frac{1}{2}(x^* - x)) = -x'(\frac{1}{2}(x - x^*)) .$$

Hence

$$x'(\frac{1}{2}(x - x^*)) = 0 ,$$

$$x'(x) = x'(\frac{1}{2}(x + x^*)) .$$

Thus x' is bounded on $E^\#$ and therefore it is continuous.

Take $x \in \mathrm{Re}\, E^\#$. Then $x^+, x^- \in E_+^\#$ (Theorem 4.2.2.9 b)) and

$$|x'(x)| = |x'(x^+) - x'(x^-)| \leq \sup\{x'(x^+), x'(x^-)\} \leq \sup_{x \in E_+^\#} x'(x)\,.$$

By Proposition 2.3.2.22 j),

$$\|x'\| \leq \sup_{x \in E_+^\#} x'(x)\,.$$

The reverse inequality is trivial. ■

Proposition 4.2.2.14 (0) *Let* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$. *Then the following are equivalent for every Gelfand* C^**-subalgebra* F *of* E*:*

a) F *is a maximal Gelfand* C^**-subalgebra of* E.

b) $F = F^c$ $(F = \mathrm{Re}\,(F^c))$.

a $\Rightarrow$ b. Since F is commutative, $F \subset F^c$. Take $x \in \mathrm{Re}\,(F^c)$. Then $F \cup \{x\}$ is commutative (and contained in $\mathrm{Re}\,E$), so that the C^*-subalgebra of E generated by $F \cup \{x\}$ is a Gelfand C^*-subalgebra of E (Corollary 4.1.2.3). Since F is maximal, we get $x \in F$. Hence $\mathrm{Re}\,(F^c) \subset F$ and

$$F = F^c \qquad (F = \mathrm{Re}\,(F^c))\,.$$

b $\Rightarrow$ a. Let G be a maximal Gelfand C^*-subalgebra of E containing F (Corollary 4.1.2.3). By a $\Rightarrow$ b,

$$G = G^c \qquad (G = \mathrm{Re}\,(G^c))\,.$$

By b),

$$F \subset G = G^c \subset F^c = F$$

$$(F \subset G = \mathrm{Re}(G^c) \subset \mathrm{Re}(F^c) = F)\,,$$

and so $F = G$ and F is a maximal Gelfand C^*-subalgebra of E. ■

Proposition 4.2.2.15 (0) *Let* E *be a real* C^**-algebra and* $\overset{\circ}{E}$ *its complexification.*

a) *For all* (x_1, y_1), $(x_2, y_2) \in \overset{\circ}{E}$,

$$(x_1, y_1) \leq (x_2, y_2) \Longrightarrow x_1 \leq x_2\,,\ (x_1, y_2) \leq (x_2, y_1)\,.$$

b) *For all* $x \in E$,

$$x \geq 0 \Longleftrightarrow (x, 0) \geq 0\,,$$

$$(0, x) \geq 0 \Longrightarrow x = 0\,.$$

c) $\alpha \in [-1, 1]$, $(x, y) \in \overset{\circ}{E}_+ \Rightarrow (x, \alpha y) \in \overset{\circ}{E}_+$.

d) *Let* A *be a subset of* E *such that* $A \times \{0\}$ *has an infimum* (x, y) *in* $\overset{\circ}{E}$. *Then* $y = 0$ *and* x *is the infimum of* A *in* E.

e) *If* $\overset{\circ}{E}$ *is order complete (order* σ*-complete) then* E *is also order complete (order* σ*-complete).*

a) We have

$$(x_2 - x_1\,,\; y_2 - y_1) \geq 0\,.$$

By Theorem 4.2.2.1 a $\Rightarrow$ c, there is an $(x, y) \in \overset{\circ}{E}$ such that

$$(x_2 - x_1, y_2 - y_1) = (x, y)^*(x, y)\,.$$

Thus

$$x_2 - x_1 = x^*x + y^*y\,, \quad y_2 - y_1 = x^*y - y^*x\,,$$

$$(x_2, y_1) - (x_1, y_2) = (x_2 - x_1, y_1 - y_2) =$$

$$= (x^*x + y^*y\,,\; -x^*y + y^*x) = (x, -y)^*(x, -y)\,.$$

By Theorem 4.2.2.1 c $\Rightarrow$ a,

$$x_1 \leq x_2\,, \quad (x_1, y_2) \leq (x_2, y_1)\,.$$

b) Assume that $x \geq 0$. There is a $y \in E$ with $y^*y = x$. Then

$$(x, 0) = (y^*y, 0) = (y, 0)^*(y, 0) \geq 0\,.$$

The implications

$$(x, 0) \geq 0 \Longrightarrow x \geq 0\,,$$

$$(0, x) \geq 0 \Longrightarrow x = 0$$

follow from a).

c) By a),

$$(0,y) \leq (x,0)\,,$$

$$0 \leq (x,-y)\,.$$

Let $\beta, \gamma \in [0,1]$ such that

$$\beta + \gamma = 1\,, \quad \beta - \gamma = \alpha\,.$$

Then

$$0 \leq \beta(x,y) + \gamma(x,-y) = ((\beta+\gamma)x, (\beta-\gamma)y) = (x, \alpha y)\,.$$

d) By a), x is a lower bound for A in E. Let z be a lower bound for A in E. By b), $(z,0)$ is a lower bound for $A \times \{0\}$ in $\overset{\circ}{E}$. Hence

$$(z,0) \leq (x,y)\,.$$

By a), $z \leq x$, so that x is the infimum of A in E. $(0,y)$ is the infimum of $(A-x) \times \{0\}$ in $\overset{\circ}{E}$. By b), 0 is a lower bound for $(A-x) \times \{0\}$ in $\overset{\circ}{E}$. Hence

$$(0,y) \geq 0$$

and therefore $y = 0$ by b).

e) follows from d). ■

Proposition 4.2.2.16 (0) *Let E be unital. Take $B \subset A \subset E$. Let x be the infimum of B in A and y an invertible element of E such that $y^*Ay = A$. Then y^*xy is the infimum of y^*By in A.*

We have

$$y^*xy \leq y^*zy$$

for every $z \in B$ (Corollary 4.2.2.3), so y^*xy is a lower bound for y^*By in A. Let x_0 be a lower bound for y^*By in A. Then

$$x_0 \leq y^*zy\,,$$

$$(y^{-1})^*x_0y^{-1} \leq z$$

for every $z \in B$ (Corollary 4.2.2.3), so that

$$(y^{-1})^*x_0y^{-1} \leq x$$

$$x_0 \leq y^*xy\,.$$

Hence y^*xy is the infimum of y^*By in A. ■

Proposition 4.2.2.17 (0) *Let E be unital, B a downward directed set of E, and x the infimum of B in E. If one of the assumptions from below is fulfilled then $B^c \subset \{x\}^c$:*

a) $\mathbb{K} = \mathbb{C}$;

b) $\mathbb{K} = \mathbb{R}$ *and* $\overset{\circ}{E}$ *is order complete;*

c) $\mathbb{K} = \mathbb{R}$, $\overset{\circ}{E}$ *is order σ-complete, and B is countable.*

a) Let y be a unitary element of B^c. Then

$$y^*zy = y^*yz = z$$

for all $z \in B$, so that $y^*By = B$. By Proposition 4.2.2.16,

$$y^*xy = x\,.$$

Hence

$$xy = yx\,,$$

$$y \in \{x\}^c\,.$$

Since every element of B^c is the linear combination of four unitary elements of B^c (Corollary 4.1.3.7, Corollary 4.1.4.2 b)) we deduce

$$B^c \subset \{x\}^c\,.$$

b) or c). By Proposition 4.2.2.15 d), $(x,0)$ is the infimum of $B \times \{0\}$ in $\overset{\circ}{E}$. By a),

$$(B \times \{0\})^c \subset \{(x,0)\}^c\,.$$

It follows that

$$B^c \subset \{x\}^c\,. \qquad \blacksquare$$

Definition 4.2.2.18 (0) *A subset A of E is called* ***order faithful*** *if for every downward (upward) directed subset of A, which has an infimum (supremum) in E, the infimum (supremum) belongs to A. A is called* ***order σ-faithful*** *if the countable subsets of A have the last property.*

If E is order complete (order σ-complete), then every order faithful (order σ-faithful) set of E is order complete (order σ-complete).

Corollary 4.2.2.19 (0) *Let A be a subset of E.*

a) *If $\mathbb{K} = \mathbb{C}$ or if $\mathbb{K} = \mathbb{R}$ and $\overset{\circ}{E}$ is order complete then A^c is order faithful.*

b) *If $\mathbb{K} = \mathbb{R}$ and $\overset{\circ}{E}$ is order σ-complete then A^c is order σ-faithful.*

a) (resp. b)) Let B be a downward directed (resp. countable) subset of A^c and let x be its infimum in E. By Proposition 4.2.2.17 and Proposition 2.1.1.17 b),c),

$$A \subset A^{cc} \subset B^c \subset \{x\}^c .$$

Hence $x \in A^c$ and A^c is order faithful (resp. order σ-faithful). ■

Corollary 4.2.2.20 (0) *Let F be a maximal Gelfand C^*-subalgebra of E.*

a) *If $\mathbb{K} = \mathbb{C}$ or if $\mathbb{K} = \mathbb{R}$ and $\overset{\circ}{E}$ is order complete then F is order faithful. If in addition E is order complete (order σ-complete) then F is also order complete (order σ-complete) and $\sigma(F)$ is a Stone space (a σ-Stone space).*

b) *If $\mathbb{K} = \mathbb{R}$ and $\overset{\circ}{E}$ is order σ-complete then F is order σ-faithful, F is order σ-complete, and $\sigma(F)$ is a σ-Stone space.*

a) (resp. b)). By Proposition 4.2.2.14 a $\Rightarrow$ b, $F = F^c$ if $\mathbb{K} = \mathbb{C}$ and $F = \mathrm{Re}\,(F^c)$ if $\mathbb{K} = \mathbb{R}$. By Corollary 4.2.2.19, F is order faithful (resp. order σ-faithful). If $\overset{\circ}{E}$ is order complete (resp. order σ-complete), then by Proposition 4.2.2.15 e), E is order complete (resp. order σ-complete). It follows that F is order complete (order σ-complete). By Proposition 1.7.2.13 e $\Rightarrow$ a, $\sigma(F)$ is a Stone space (σ-Stone space). ■

Remark. Let F be a maximal Gelfand C^*-subalgebra and A a subset of F The above proof shows that under the above hypotheses if A has an infimum x in E then $x \in F$, even if A is not downward directed. But it may happen that A has an infimum in F and no infimum in E, as the following example shows. $E := \mathcal{L}(\mathbb{K}^2)$,

$$F := \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{K} \right\} ,$$

$$A := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Proposition 4.2.2.21 (0) *Let E be unital and A a downward directed set of E (of E_+). Let x be the infimum of A in E (in E_+). Then for every $y \in E$, y^*xy is the infimum of y^*Ay in E (in E_+) (Corollary 4.2.2.3). In particular, pEp is an order faithful C^*-subalgebra of E for every $p \in \Pr E$.*

Let F be the set of all $y \in E$ for which y^*xy is the infimum of y^*Ay in E (in E_+). Take $y, z \in F$ and let b be a lower bound for $(y+z)^*A(y+z)$ in E (in E_+). Take $a_1, a_2 \in A$. There is an $a \in A$ with

$$a \leq a_1, \quad a \leq a_2.$$

Then

$$b \leq (y+z)^*a(y+z),$$

$$b - (y+z)^*x(y+z) \leq (y+z)^*(a-x)(y+z) =$$

$$= y^*(a-x)y + z^*(a-x)z + y^*(a-x)z + z^*(a-x)y \leq$$

$$\leq y^*(a-x)y + z^*(a-x)z + y^*(a-x)y + z^*(a-x)z =$$

$$= 2(y^*(a-x)y + z^*(a-x)z)$$

(Corollary 4.2.2.4),

$$\frac{1}{2}(b - (y+z)^*x(y+z)) \leq y^*(a-x)y + z^*(a-x)z \leq$$

$$\leq y^*(a_1 - x)y + z^*(a_2 - x)z = y^*a_1y - y^*xy + z^*a_2z - z^*xz$$

(Corollary 4.2.2.3). Since a_1 and a_2 are arbitrary and $y, z \in F$,

$$\frac{1}{2}(b - (y+z)^*x(y+z)) \leq 0,$$

$$b \leq (y+z)^*x(y+z).$$

Hence $(y+z)^*x(y+z)$ is the infimum of $(y+z)^*A(y+z)$ in E (in E_+), $y+z \in F$, and F is a vector subspace of E. By Proposition 4.2.2.16, F contains all invertible elements of E. Since every element of E is the difference of two invertible elements of E, it follows that $F = E$. ∎

Corollary 4.2.2.22 (7) *Let E be unital, A a downward directed set of E_+, and x the infimum of A in E_+. Then x is the infimum of A in E.*

Let y be a lower bound for A in E. Then $y \in \operatorname{Re} E$, $y - x$ is a lower bound for $A - x$ in E and 0 is the infimum for $A - x$ in E_+. By Proposition 4.2.2.21, 0 is the infimum of $(y-x)^+(A-x)(y-x)^+$ in E_+. It follows from

$$((x-y)^+)^3 = (y-x)^+(y-x)(y-x)^+$$

that $((y-x)^+)^3$ is a lower bound for $(y-x)^+(A-x)(y-x)^+$ in E_+. Hence

$$((y-x)^+)^3 = 0\,,$$

$$(y-x)^+ = 0\,,$$

$$y - x = -(y-x)^- \leq 0\,,$$

$$y \leq x\,,$$

i.e. x is the infimum of A in E. ■

Proposition 4.2.2.23 (0) *Take $x, y, z \in E$ with*

$$x^*x \leq y^*y\,, \quad y^*yz = 0 \quad (xx^* \leq yy^*\,,\ zyy^* = 0)\,.$$

Then

$$xz = 0 \quad (zx = 0)\,.$$

We have that

$$0 \leq z^*x^*xz \leq z^*y^*yz = 0 \quad (0 \leq zxx^*z^* \leq zyy^*z^* = 0)$$

(Corollary 4.2.2.3),

$$z^*x^*xz = 0 \quad (zxx^*z^* = 0)\,,$$

$$\|xz\|^2 = \|z^*x^*xz\| = 0 \quad (\|zx\|^2 = \|zxx^*z^*\| = 0)\,,$$

$$xz = 0 \quad (zx = 0)\,.$$ ■

Proposition 4.2.2.24 *Let E be a complex C^*-algebra and $\overset{\circ}{E}$ the complexification of the underlying involutive real algebra of E. The element $(x,y) \in \overset{\circ}{E}$ is positive (Theorem 4.1.1.8) iff $x+iy$ and $x-iy$ are positive.*

Assume that (x,y) is positive. Then there is an $(a,b) \in \overset{\circ}{E}$ such that

$$(x,y) = (a,b)^*(a,b)$$

(Theorem 4.2.2.1 a $\Rightarrow$ c). Thus

$$x = a^*a + b^*b\,, \quad y = a^*b - b^*a\,,$$

$$(a+ib)^*(a+ib) = (a^* - ib^*)(a+ib) =$$

$$= a^*a + b^*b + i(a^*b - b^*a) = x + iy\,,$$

$$(a-ib)^*(a-ib) = (a^* + ib^*)(a-ib) =$$

$$= a^*a + b^*b - i(a^*b - b^*a) = x - iy\,.$$

Hence $x+iy$ and $x-iy$ are positive (Theorem 4.2.2.1 c $\Rightarrow$ a).

For the converse, assume that $x+iy$ and $x-iy$ are positive. Then there are $c,d \in E$ with

$$x + iy = c^*c\,, \quad x - iy = d^*d$$

(Theorem 4.2.2.1 a $\Rightarrow$ c). We have

$$\left(\frac{1}{2}(c+d), \frac{1}{2i}(c-d)\right)^* \left(\frac{1}{2}(c+d), \frac{1}{2i}(c-d)\right) =$$

$$= \frac{1}{4}\left(c^* + d^*\,, \frac{1}{i}(c^* - d^*)\right)\left(c+d\,, \frac{1}{i}(c-d)\right) =$$

$$= \frac{1}{4}\begin{pmatrix} (c^*+d^*)(c+d) + (c^*-d^*)(c-d)\,, \\ \frac{1}{i}((c^*+d^*)(c-d) + (c^*-d^*)(c+d)) \end{pmatrix} =$$

$$= \frac{1}{4}\left(2c^*c + 2d^*d\,, \frac{1}{i}(2c^*c - 2d^*d)\right) = (x,y)\,.$$

Hence (x,y) is positive (Theorem 4.2.2.1 c $\Rightarrow$ a). ∎

4.2.3 Examples

Example 4.2.3.1 *Let T be a set. An element x of the C^*-algebra $\ell^\infty(T)$ (Example 4.1.1.5) is positive iff*

$$x(t) \in \mathbb{R}_+$$

for every $t \in T$.

We have

$$\sigma(x) = \overline{x(T)}$$

(Example 2.1.4.3). ■

Example 4.2.3.2 *Take $n \in \mathbb{N}$ and let $\mathcal{L}(\mathbb{K}^n)$ be the C^*-algebra defined in Example 4.1.1.7. Define*

$$\widetilde{x} : \mathcal{L}(\mathbb{K}^n) \longrightarrow \mathbb{K}, \quad a \longmapsto \sum_{i,j=1}^{n} a_{ij}x_{ji}$$

for every $x \in \mathcal{L}(\mathbb{K}^n)$, where we have identified the elements of $\mathcal{L}(\mathbb{K}^n)$ with their associated $n \times n$-matrices. Then $\widetilde{x} \in \mathcal{L}(\mathbb{K}^n)'$ for every $x \in \mathcal{L}(\mathbb{K}^n)$, the map

$$\mathcal{L}(\mathbb{K}^n) \longrightarrow \mathcal{L}(\mathbb{K}^n)', \quad x \longmapsto \widetilde{x}$$

is bijective, linear, and involutive and for every $x \in \mathcal{L}(\mathbb{K}^n)$ the following are equivalent:

a) $x \geq 0$.

b) $\widetilde{x} \geq 0$.

The first two assertions were proved in Example 2.3.1.33.

a $\Rightarrow$ b. By Theorem 4.2.2.1 a $\Rightarrow$ c (and Example 4.1.1.7), there is a $y \in \mathbb{K}_{n,n}$ with

$$x = y^*y.$$

Thus

$$\widetilde{x}(a^*a) = \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} a_{ik}^* a_{kj} \right) \left(\sum_{\ell=1}^{n} y_{j\ell}^* y_{\ell i} \right) =$$

$$= \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \overline{a}_{ki} a_{kj} \right) \left(\sum_{\ell=1}^{n} \overline{y}_{\ell j} y_{\ell i} \right) =$$

$$= \sum_{k,\ell=1}^{n} \left(\sum_{i=1}^{n} \overline{a}_{ki} y_{\ell i} \right) \left(\sum_{j=1}^{n} a_{kj} \overline{y}_{\ell j} \right) = \sum_{k,\ell=1}^{n} \left| \sum_{i=1}^{n} \overline{a}_{ki} y_{\ell i} \right|^2 \in \mathbb{R}_+$$

for every $a \in \mathbb{K}_{n,n}$. Hence $\widetilde{x}$ is positive.

b $\Rightarrow$ a. By Example 2.3.1.33, $x \in \operatorname{Re} \mathbb{K}_{n,n}$. By Example 2.3.4.3, $\sigma(x) \subset \mathbb{R}_+$. ■

Proposition 4.2.3.3 *Take $n \in \mathbb{N}$ and $[a_{ij}] \in \mathbb{R}_{n,n}$. Then the following are equivalent:*

a) *The linear form*

$$\mathbb{R}_{n,n} \longrightarrow \mathbb{R}, \quad [\alpha_{ij}] \longmapsto \sum_{i,j=1}^{n} \alpha_{ij} a_{ji}$$

is positive (Example 2.3.1.32).

b) *The linear form*

$$\mathbb{C}_{n,n} \longrightarrow \mathbb{C}, \quad [\alpha_{ij}] \longmapsto \sum_{1,j=1}^{n} \alpha_{ij} a_{ji}$$

is positive (Example 2.3.1.32).

a $\Rightarrow$ b. $[a_{ij}]$ is selfadjoint and by the last assertion of Example 2.3.4.3, $\sigma([a_{ij}]) \subset \mathbb{R}_+$. b) follows now from Example 4.2.3.2 a $\Rightarrow$ b.

b $\Rightarrow$ a is trivial. ■

Example 4.2.3.4 *Take $n \in \mathbb{N}$ and let E be the C^*-algebra $\mathcal{L}(\mathbb{K}^n)$ (Example 4.1.1.7). Given $x, y \in \mathbb{K}^n$ and $a \in E$, define*

$$\langle x|y \rangle := \sum_{i=1}^{n} x_i \overline{y}_i,$$

$$[x|y] := [x_i \overline{y}_j]_{i,j \in \mathbb{N}_n} \in E$$

and

$$\widetilde{a} : E \longrightarrow \mathbb{K}, \quad [\alpha_{ij}]_{i,j \in \mathbb{N}_n} \longmapsto \sum_{i,j=1}^{n} \alpha_{ij} a_{ji}.$$

Then, given $a \in E$:

a) $\widetilde{a} \in \tau(E)$ *iff there exist a finite subset* A *of* $\mathbb{K}^n$ *and a function* $f : A \to \mathbb{R}_+$ *such that*

$$\langle x|y\rangle = \delta_{x,y}$$

for all $x, y \in A$ *and*

$$a = \sum_{x \in A} f(x)[x|x]\,, \quad \sum_{x \in A} f(x) = 1\,.$$

b) $\widetilde{a} \in \tau_0(E)$ *iff there is an* $x \in \mathbb{K}^n$ *such that*

$$a = [x|x]\,, \quad \|x\| = 1\,.$$

a) By Proposition 4.2.3.3. a $\Rightarrow$ b, it suffices to prove the statement for the complex case. By Example 4.2.3.2, a is a positive element of $\mathcal{L}(\mathbb{C}^n)$ iff $\widetilde{a}$ is positive. Hence identifying $\mathcal{L}(\mathbb{C}^n)$ with $\mathcal{L}(\mathbb{C}^n)'$ via the map

$$\mathcal{L}(\mathbb{C}^n) \longrightarrow \mathcal{L}(\mathbb{C}^n)', \quad a \longmapsto \widetilde{a}\,,$$

it follows

$$\tau(\mathcal{L}(\mathbb{C}^n)) = \{a \in \mathcal{L}(\mathbb{C}^n)_+ \mid \|\widetilde{a}\| = 1\}\,.$$

Take $\widetilde{a} \in \tau(\mathcal{L}(\mathbb{C}^n))$. By Example 4.1.4.11, there are a finite subset A of $\mathbb{C}^n$ and a function $f : A \to \mathbb{R}_+$ such that

$$\langle x|y\rangle = \delta_{x,y}$$

for all $x, y \in A$ and

$$a = \sum_{x \in A} f(x)[x|x]\,,$$

$$\sum_{x \in A} f(x) = \|\widetilde{a}\| = 1\,.$$

The converse is trivial.

b) Take $x \in \mathbb{C}^n$ (resp. $x \in \mathbb{R}^n$) with $\|x\| = 1$. Then $\widetilde{[x|x]} \in \tau(E)$ by a). Take $a, b \in E$ and $\alpha, \beta \in]0,1[$ such that

$$\widetilde{a}, \widetilde{b} \in \tau(E)\,, \quad \alpha + \beta = 1\,, \quad [x|x] = \alpha a + \beta b\,.$$

By a), there are finite subsets A, B of $\mathbb{K}^n$ and functions $f : A \to]0, \infty[$, $g : B \to]0, \infty[$, such that

$$\langle y|z\rangle = \delta_{y,z}$$

for all $y, z \in A$ and all $y, z \in B$ and

$$a = \sum_{y\in A} f(y)[y|y]\,, \quad \sum_{y\in A} f(y) = 1\,,$$

$$b = \sum_{z\in B} g(z)[z|z]\,, \quad \sum_{z\in B} g(z) = 1\,.$$

Thus

$$[x|x] = \alpha \sum_{y\in A} f(y)[y|y] + \beta \sum_{z\in A} g(z)[z|z]\,,$$

and

$$1 = \langle [x|x]\,,\, \widetilde{[x|x]}\rangle =$$

$$= \alpha \sum_{y\in A} f(y)\langle [y|y]\,,\, \widetilde{[x|x]}\rangle + \beta \sum_{z\in\beta} g(z)\langle [z|z]\,,\, \widetilde{[x|x]}\rangle =$$

$$= \alpha \sum_{y\in A} f(y)|\langle x|y\rangle|^2 + \beta \sum_{z\in B} g(z)|\langle x|z\rangle|^2 \leq$$

$$\leq \alpha \sum_{y\in A} f(y) + \beta \sum_{z\in B} g(z) = 1$$

by Schwarz's Inequality. Hence

$$|\langle x|y\rangle| = 1$$

for all $y \in A \cup B$. This implies that A and B contain exactly one element and

$$a = b = [x|x]\,.$$

Thus $[x|x] \in \tau_0(E)$.

The converse follows from a). ∎

Example 4.2.3.5 *The matrix*

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathcal{L}(\mathbb{K}^2)$$

is positive (Example 4.1.1.7) iff

$$\alpha, \delta \in \mathbb{R}_+\,, \quad \beta = \overline{\gamma}\,, \quad |\beta|^2 \leq \alpha\delta\,.$$

This follows from Example 4.2.3.2 and Example 2.3.4.4 b $\Leftrightarrow$ c. ■

Remark. a) Put

$$x := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad y := \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad z := \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then x, y, z are positive and

$$x \leq y + z,$$

but there are no positive matrices x_1, x_2 for which

$$x = x_1 + x_2, \quad x_1 \leq y, \quad x_2 \leq z.$$

In particular, $\operatorname{Re}\mathcal{L}(\mathbb{K}^2)$ is not a vector lattice.

b) The assertion of the above example still holds for $\mathbb{H}_{2,2}$.

Example 4.2.3.6 *Put*

$$x := \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix} \in \operatorname{Re}\mathcal{L}(\mathbb{K}^2), \quad y := \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix} \in \operatorname{Re}\mathcal{L}(\mathbb{K}^2)$$

(Example 4.1.1.7).

a) $|\alpha| \leq \|x\|, \quad |\delta| \leq \|x\|, \quad |\beta|^2 \leq (\|x\| \pm \alpha)(\|x\| \pm \delta)$.

b) $(x \in \mathcal{L}(\mathbb{K}^2)_+, \|x\| < 1) \Leftrightarrow$

$$\Leftrightarrow (\alpha, \delta \in [0, 1[, |\beta|^2 \leq \alpha\delta, |\beta|^2 < (1-\alpha)(1-\delta)).$$

c) $y \in \mathcal{L}(\mathbb{K}^2)_+, \quad \|y\| < 1,$ *and there is no* $\gamma \in \mathbb{R}_+$ *with*

$$y \leq \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix}.$$

a) By Corollary 4.2.1.16 a $\Rightarrow$ b,

$$\begin{bmatrix} \|x\| \pm \alpha & \pm\beta \\ \pm\overline{\beta} & \|x\| \pm \delta \end{bmatrix} = \|x\| 1 \pm x \in \mathcal{L}(\mathbb{K}^2)_+,$$

so that, by Example 4.2.3.5,

$$\|x\| \pm \alpha, \|x\| \pm \delta \in \mathbb{R}_+, \quad |\beta|^2 \leq (\|x\| \pm \alpha)(\|x\| \pm \delta).$$

b) follows from a), Example 4.2.3.5, and Corollary 4.2.1.16 b $\Rightarrow$ a.

c) follows from b) and Example 4.2.3.5. ■

Example 4.2.3.7 *Let*

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix}$$

be a selfadjoint element of $\mathcal{L}(\mathbb{K}^2)$ *(Example 4.1.1.7) which is neither positive nor negative. Define*

$$\rho := \sqrt{(\alpha-\delta)^2 + 4|\beta|^2} \in \mathbb{R}_+ ,$$

$$a := \frac{\alpha+\delta+\rho}{2}, \quad b := \frac{\alpha+\delta-\rho}{2} .$$

Then $\rho > 0$,

$$b < 0 < a ,$$

and

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix}^+ = \frac{a}{\rho}\begin{bmatrix} \frac{\alpha-\delta+\rho}{2} & \beta \\ \overline{\beta} & \frac{\delta-\alpha+\rho}{2} \end{bmatrix} = \frac{a}{\rho}\begin{bmatrix} \frac{\alpha-\delta}{2} & \beta \\ \overline{\beta} & \frac{\delta-\alpha}{2} \end{bmatrix} + \frac{a}{2}1 ,$$

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix}^- = \frac{b}{\rho}\begin{bmatrix} \frac{\alpha-\delta-\rho}{2} & \beta \\ \overline{\beta} & \frac{\delta-\alpha-\rho}{2} \end{bmatrix} = \frac{b}{\rho}\begin{bmatrix} \frac{\alpha-\delta}{2} & \beta \\ \overline{\beta} & \frac{\delta-\alpha}{2} \end{bmatrix} - \frac{b}{2}1 .$$

a and b are the zeros of the polynomial

$$\begin{bmatrix} \alpha - t & \beta \\ \overline{\beta} & \delta - t \end{bmatrix} \in \mathbb{K}[t] ,$$

so that the hypotheses imply

$$b < 0 < a , \quad \rho > 0 .$$

The second claim follows immediately from Example 4.1.3.19 and Theorem 4.2.2.9 b). ■

Remark. Let E be a C^*–algebra. Take $x, y \in E_+$ and put

$$z = x - y .$$

If $xy = yx$ then

$$z^+ \leq x\,, \quad z^- \leq x\,,$$

but these inequalities do not hold in general without the above commutativity condition as the example

$$x := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\,, \quad y := \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

shows.

Example 4.2.3.8 *If $(T, \mathfrak{T}, \mu)$ is a measure space, then the order relation on the C^*–algebra $L^\infty(\mu)$ (Example 4.1.1.5) coincides with the usual order relation on $L^\infty(\mu)$.*

Take $x \in L^\infty(\mu)$. By Example 2.2.2.2,

$$\sigma(x) \subset \mathbb{R}_+ \Longleftrightarrow x \geq 0 \quad \mu\text{–a.e.}$$ ■

Proposition 4.2.3.9 (0) *If $x \in \operatorname{Re} E$ then the following are equvalent:*

a) $x^2 \in \operatorname{Pr} E$.

b) $|x| \in \operatorname{Pr} E$.

c) $x^2 = |x|$.

If these conditions are fulfilled then

$$x^+ = \frac{1}{2}(|x| + x) \in \operatorname{Pr} E\,, \quad x^- = \frac{1}{2}(|x| - x) \in \operatorname{Pr} E\,,$$

$$x^+ x = x x^+ = x^+\,, \quad x^- x = x x^- = -x^-\,.$$

$E(x)$ is a Gelfand C^*–subalgebra of E (Corollary 4.1.2.3) and the assertions follow by Gelfand transform. ■

4.2.4 Powers of Positive Elements

Definition 4.2.4.1 (0) *Take* $p > 0$ *and*

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ , \quad \alpha \longmapsto \alpha^p .$$

Given $x \in E_+$, *define*

$$x^p := f(x) .$$

By Theorem 4.1.3.1 a), the above definition coincides with Definition 2.1.1.1 whenever $p \in \mathbb{N}$.

Proposition 4.2.4.2 (0) *Take* $\alpha, \beta \in \mathbb{R}_+ \backslash \{0\}$.

a) *The map*

$$E_+ \longrightarrow E_+ , \quad x \longmapsto x^\alpha$$

is a homeomorphism.

b) *If* $x \in E_+$, *then* $xy \in E_+$,

$$x^{\alpha+\beta} = x^\alpha x^\beta , \quad x^{\alpha\beta} = (x^\alpha)^\beta ,$$

$$\|x^\alpha\| = \|x\|^\alpha , \quad \{x\}^c = \{x^\alpha\}^c , \quad E(x) = E(x^\alpha) .$$

c) *If* $x, y \in E_+$ *and* $xy = yx$, *then* $xy \in E_+$,

$$(xy)^\alpha = x^\alpha y^\alpha ,$$

$$x \leq y \Longleftrightarrow x^\alpha \leq y^\alpha .$$

d) *If* $x \in E_+$, *then* x^α *belongs to the closed real subalgebra of* E *generated by* x .

e) $x \in E_+^\#$, $0 < \alpha < \beta \Rightarrow x^\beta \leq x^\alpha$.

f) *Take* $x \in E_+^\#$ *and assume either* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{K} = \mathbb{R}$ *and* $\overset{\circ}{E}$ *order* σ*-complete. If* $(x^{\frac{1}{n}})_{n\in\mathbb{N}}$ *has a supremum* p *in* E , *then* $p \in \Pr E$.

g) $x \in E_+^c$, $x' \in E_+'$ $\Rightarrow xx' = x'x \in E_+'$.

h) *Let F be an E-module, $x \in E_+$, and $a \in F$ such that $ax = 0$ (resp. $xa = 0$). Then*

$$ax^\alpha = 0 \qquad (resp.\ x^\alpha a = 0)\,.$$

a) follows from Proposition 4.1.4.5.

b) Put

$$F := E(x)\,.$$

Then

$$\widehat{x^{(\alpha+\beta)}} = \left(\widehat{x}\right)^{\alpha+\beta} = \widehat{(x)}^\alpha \widehat{(x)}^\beta = \widehat{x^\alpha x^\beta}\,,$$

$$\widehat{x^{\alpha\beta}} = \left(\widehat{x}\right)^{\alpha\beta} = \left(\left(\widehat{x}\right)^\alpha\right)^\beta = \left(\widehat{(x^\alpha)}\right)^\beta = \widehat{(x^\alpha)^\beta}$$

(Corollary 4.1.3.5), so that

$$x^{\alpha+\beta} = x^\alpha x^\beta\,, \quad x^{\alpha\beta} = (x^\alpha)^\beta$$

(Corollary 4.1.2.5). The relations

$$\|x^\alpha\| = \|x\|^\alpha\,, \quad \{x\}^c = \{x^\alpha\}^c\,, \quad E(x) = E(x^\alpha)$$

follow from Corollary 4.2.1.7 and Proposition 4.1.4.5.

c) follows from the Gelfand transform.

d) follows from Corollary 4.1.3.3 a).

e) Since

$$\sigma(x) \subset [0,1]$$

(Corollary 4.2.1.16 a $\Rightarrow$ c), the claim follows from Corollary 4.2.1.12.

f) Let F be a maximal Gelfand C^*-subalgebra of E containing

$$\{x^{\frac{1}{n}} \mid n \in \mathbb{N}\}$$

(b) and Corollary 4.1.2.3). By Corollary 4.2.2.20, $p \in F$. Using the Gelfand transform on F, it is easy to see that p is an orthogonal projection in E.

g) By Proposition 2.2.7.4 a),

$$xx' = x'x\,,$$

so that

$$(xx')^* = x'x = xx' ,$$

$$xx' \in \operatorname{Re} E' .$$

Take $y \in E_+$. By c), $yx \in E_+$ and so

$$\langle y, xx' \rangle = \langle yx, x' \rangle \in \mathbb{R}_+ ,$$

$xx' \in E'_+$ (Corollary 4.2.2.10).

h) follows from Theorem 4.1.3.1 f). ∎

Corollary 4.2.4.3 (0) *Given $x \in E_+$, the following are equivalent:*

a) *there are distinct $\alpha, \beta \in \mathbb{R}_+ \backslash \{0\}$, with $x^\alpha = x^\beta$.*

b) *$x^\alpha = x$ for every $\alpha \in \mathbb{R}_+ \backslash \{0\}$.*

c) *$x \in \Pr E$.*

a $\Rightarrow$ b. Define

$$f : \sigma(x) \longrightarrow \mathbb{R} , \quad \gamma \longmapsto \gamma^\alpha - \gamma^\beta .$$

Then

$$f(x) = 0 ,$$

so that

$$f(\sigma(x)) = \sigma(f(x)) = \{0\}$$

(Corollary 4.1.3.5),

$$\sigma(x) \subset \{0, 1\} .$$

b) now follows.

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By Proposition 4.1.2.21 a $\Rightarrow$ c,

$$\sigma(x) \subset \{0, 1\}$$

and a) now follows. ∎

Definition 4.2.4.4 (0) *Let E be unital. Take $p \in \mathbb{R}$ and*

$$f : \mathbb{R}_+\backslash\{0\} \longrightarrow \mathbb{R}_+\backslash\{0\}\,, \quad \alpha \longmapsto \alpha^p\,,$$

$$g : \mathbb{R}_+\backslash\{0\} \longrightarrow \mathbb{R}\,, \quad \alpha \longmapsto \log \alpha\,.$$

Given $x \in E_+$, if x is invertible, define

$$x^p := f(x)\,,$$

$$\log x := g(x)\,.$$

If $p > 0$, then the above definition of x^p coincides with Definition 4.2.4.1. By Theorem 4.1.3.1 a), the above definition of x^p coincides with Definition 2.1.2.5 whenever $-p \in \mathbb{N}$. Since $x^0 = 1$, the above Definition is compatible with Definition 2.1.1.3.

By Corollary 4.1.3.9, for every $x \in U_1^E(0)$, $(1+x)^\alpha$ and $\log(1-x)$ defined above coincide with $[1+x]^\alpha$ and $\log(1-x)$ defined in Proposition 2.2.3.13 a) and 2.2.3.9, respectively.

Proposition 4.2.4.5 (0) *Let E be unital. Take $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, and put*

$$A := \{x \in E_+ \mid x \text{ is invertible}\}\,.$$

a) *The maps*

$$A \longrightarrow A\,, \quad x \longmapsto x^\alpha\,,$$

$$A \longrightarrow \operatorname{Re} E\,, \quad x \longmapsto \log x\,,$$

are homeomorphisms.

b) *If $x \in A$ then*

$$x^{\alpha+\beta} = x^\alpha x^\beta\,, \quad x^{\alpha\beta} = (x^\alpha)^\beta\,,$$

$$e^{\log x} = x\,, \quad \log x^\alpha = \alpha \log x\,, \quad \log x \leq x\,,$$

$$\{x\}^c = \{x^\alpha\}^c = \{\log x\}^c\,, \quad E(x) = E(x^\alpha) = E(x,1) = E(\log x, 1)\,.$$

c) *Given* $x, y \in A$ *with* $xy = yx$,

$$(xy)^\alpha = x^\alpha y^\alpha, \quad \log(xy) = \log x + \log y,$$

$$x \leq y \Longleftrightarrow \log x \leq \log y.$$

a) follows from Proposition 4.1.4.5 and Corollary 4.1.2.13 b $\Leftrightarrow$ a.

b) Define

$$F := E(x, 1).$$

Then

$$\overset{F}{\widehat{x^{(\alpha+\beta)}}} = \left(\overset{F}{\widehat{x}}\right)^{\alpha+\beta} = \overset{F}{\widehat{(x)}}^{\alpha} \overset{F}{\widehat{(x)}}^{\beta} = \overset{F}{\widehat{x^\alpha x^\beta}},$$

$$\overset{F}{\widehat{x^{\alpha\beta}}} = \left(\overset{F}{\widehat{x}}\right)^{\alpha\beta} = \left(\left(\overset{F}{\widehat{x}}\right)^{\alpha}\right)^{\beta} = \left(\overset{F}{\widehat{(x^\alpha)}}\right)^{\beta} = \overset{F}{\widehat{(x^\alpha)^\beta}},$$

$$\overset{F}{\widehat{\log x^\alpha}} = \log \overset{F}{\widehat{x^\alpha}} = \log \left(\overset{F}{\widehat{x}}\right)^{\alpha} = \alpha \log \left(\overset{F}{\widehat{x}}\right) = \alpha \overset{F}{\widehat{\log x}},$$

(Corollary 4.1.3.5), so that

$$x^{\alpha+\beta} = x^\alpha x^\beta, \quad x^{\alpha\beta} = (x^\alpha)^\beta, \quad \log x^\alpha = \alpha \log x$$

(Corollary 4.1.2.5). The relations

$$\log x \leq x, \quad e^{\log x} = x,$$

$$\{x\}^c = \{x^\alpha\}^c = \{\log x\}^c, \quad E(x) = E(x^\alpha) = E(x, 1) = E(\log x, 1)$$

follow from Corollary 4.2.1.12 and Proposition 4.1.4.5.

c) follows from the Gelfand transform. ∎

Proposition 4.2.4.6 (0) *Let* E *be unital and take* $x, y \in E_+$ *with* $x \leq y$. *If* x *is invertible, then* y *is also invertible and* $y^{-1} \leq x^{-1}$.

There is an $\alpha > 0$ such that $\alpha 1 \leq x$ (Proposition 4.2.1.15). Hence $\alpha 1 \leq y$ and y is invertible (Corollary 4.2.1.13). Now

$$1 = x^{-\frac{1}{2}} x x^{-\frac{1}{2}} \leq x^{-\frac{1}{2}} y x^{-\frac{1}{2}}$$

(Proposition 4.2.4.5 b), Corollary 4.2.2.3). Thus

$$x^{\frac{1}{2}} y^{-1} x^{\frac{1}{2}} = \left(x^{-\frac{1}{2}} y x^{-\frac{1}{2}} \right)^{-1} \leq 1$$

(Corollary 4.2.1.13) and

$$y^{-1} \leq x^{-\frac{1}{2}} 1 x^{-\frac{1}{2}} = x^{-1}$$

(Proposition 4.2.4.5 b), Corollary 4.2.2.3). ■

Corollary 4.2.4.7 (0) *If E is unital, if $x, y \in E_+$ with $x \leq y$ and if $\alpha > 0$, then:*

a) $x(\alpha 1 + x)^{-1} \leq y(\alpha 1 + y)^{-1}$.

b) $0 \leq x^{\frac{1}{2}} (\alpha 1 + y)^{-1} x^{\frac{1}{2}} \leq 1$.

c) $\| x^{\frac{1}{2}} (\alpha 1 + y)^{-\frac{1}{2}} \| \leq 1$.

$\alpha 1 + x$ and $\alpha 1 + y$ are invertible and

$$(\alpha 1 + y)^{-1} \leq (\alpha 1 + x)^{-1}$$

by Proposition 4.2.4.6.

a) $x(\alpha 1 + x)^{-1} = 1 - \alpha(\alpha 1 + x)^{-1} \leq 1 - \alpha(\alpha 1 + y)^{-1} = y(\alpha 1 + y)^{-1}$.

b) By Corollary 4.2.2.3 and Corollary 4.2.1.12,

$$0 \leq x^{\frac{1}{2}} (\alpha 1 + y)^{-1} x^{\frac{1}{2}} \leq x^{\frac{1}{2}} (\alpha 1 + x)^{-1} x^{\frac{1}{2}} \leq 1 .$$

c) By b) and Corollary 4.2.1.17 b $\Rightarrow$ a,

$$\| x^{\frac{1}{2}} (\alpha 1 + y)^{-\frac{1}{2}} \|^2 = \| x^{\frac{1}{2}} (\alpha 1 + y)^{-1} x^{\frac{1}{2}} \| \leq 1 . \qquad ■$$

Proposition 4.2.4.8 (0) *Take $x, y \in E_+$. If $x \leq y$, then $x^\alpha \leq y^\alpha$ for every $\alpha \in]0, 1[$.*

Given $n \in \mathbb{N}$, define

$$f : [0, \|y\|] \longrightarrow \mathbb{R}, \quad \beta \longmapsto \int_0^\infty \frac{\beta dt}{(1 + \beta t) t^\alpha} ,$$

and

$$f_n : [0, \|y\|] \longrightarrow \mathbb{R}, \quad \beta \longmapsto \frac{1}{2^n} \sum_{k=1}^{4^n} \left(\frac{2^n}{k}\right)^{\alpha} \frac{\beta}{1 + \frac{k}{2^n}\beta}.$$

Then $f \in \mathcal{C}([0, \|y\|])$ and $(f_n)_{n \in \mathbb{N}}$ is an increasing sequence in $\mathcal{C}([0, \|y\|])$ (since the function

$$\mathbb{R}_+ \longrightarrow \mathbb{R}, \quad t \longmapsto \frac{\beta}{(1 + \beta t)t^{\alpha}}$$

is decreasing) converging pointwise to f. By Dini's Theorem,

$$\lim_{n \to \infty} \|f - f_n\| = 0.$$

Since

$$x(1 + tx)^{-1} \leq y(1 + ty)^{-1}$$

for every $t \in \mathbb{R}_+$ (Corollary 4.2.4.7 a)) we have that

$$f_n(x) = \frac{1}{2^n} \sum_{k=1}^{4^n} \left(\frac{2^n}{k}\right)^{\alpha} x \left(1 + \frac{k}{2^n}x\right)^{-1} \leq \frac{1}{2^n} \sum_{k=1}^{4^n} \left(\frac{2^n}{k}\right)^{\alpha} y \left(1 + \frac{k}{2^n}y\right)^{-1} = f_n(y)$$

for every $n \in \mathbb{N}$. Hence

$$f(x) = \lim_{n \to \infty} f_n(x) \leq \lim_{n \to \infty} f_n(y) = f(y)$$

(Theorem 4.2.1.1, Proposition 1.7.1.5 a), Theorem 4.1.3.1 b)). Since

$$f(\beta) = \int_0^{\infty} \frac{\beta dt}{(1 + \beta t)t^{\alpha}} = \int_0^{\infty} \frac{\beta^{\alpha} ds}{(1 + s)s^{\alpha}} = \left(\int_0^{\infty} \frac{ds}{(1 + s)s^{\alpha}}\right) \beta^{\alpha}$$

for every $\beta \in [0, \|y\|]$,

$$x^{\alpha} \leq y^{\alpha}.$$

■

Corollary 4.2.4.9 *Take $x, y \in \operatorname{Pr} E$. If, for some $\alpha \in \mathbb{R}_+$, $x \leq \alpha y$, then $x \leq y$.*

By Corollary 4.2.4.3 c $\Rightarrow$ b and Proposition 4.2.4.8,

$$x = x^{\beta} \leq (\alpha y)^{\beta} = \alpha^{\beta} y^{\beta} = \alpha^{\beta} y$$

for every $\beta \in]0, 1]$. Hence

$$x \leq \lim_{0 < \beta \to 0} \alpha^{\beta} y = y.$$

■

Corollary 4.2.4.10 *Take* $x, y \in \Pr E$. *If* $x = yz$ $(x = zy)$ *for some* $z \in E$, *then* $x \leq y$.

Since

$$x = x^2 = yzz^*y \leq \|z\|^2 y \quad (x = x^2 = yz^*zy \leq \|z\|^2 y)$$

(Corollary 4.2.1.16 a $\Rightarrow$ b, Corollary 4.2.2.3), $x \leq y$ by Corollary 4.2.4.9. ■

Proposition 4.2.4.11 *If for some* $\alpha \in]1, \infty[$,

$$x, y \in E_+, \, x \leq y \Longrightarrow x^\alpha \leq y^\alpha,$$

then $\operatorname{Re} E$ *is commutative.*

Step 1 $\quad x, y \in E_+, \, x \leq y \Longrightarrow x^2 \leq y^2$

By complete induction,

$$x, y \in E_+, \, x \leq y \Longrightarrow x^{\alpha^n} \leq y^{\alpha^n}$$

for every $n \in \mathbb{N}$ (Proposition 4.2.4.2 b)). By Proposition 4.2.4.8 (and Proposition 4.2.4.2 b)),

$$x, y \in E_+, \, x \leq y \Longrightarrow x^2 \leq y^2.$$

Step 2 $\quad x, y \in E_+ \Rightarrow \operatorname{re} xy \in E_+, \, (\operatorname{im} xy)^2 \leq (\operatorname{re} xy)^2$

Given $\varepsilon > 0$, it follows from Step 1 that

$$x \leq x + \varepsilon y,$$

$$x^2 \leq (x + \varepsilon y)^2 = x^2 + \varepsilon(xy + yx) + \varepsilon^2 y^2,$$

$$xy + yx + \varepsilon y^2 \geq 0.$$

Hence

$$\operatorname{re} xy = \frac{1}{2}(xy + yx) \geq 0.$$

Since yxy is positive (Corollary 4.2.2.3),

$$0 \leq \operatorname{re} x(yxy) = \operatorname{re}(xy)^2 = (\operatorname{re} xy)^2 - (\operatorname{im} xy)^2,$$

$$(\operatorname{im} xy)^2 \leq (\operatorname{re} xy)^2 .$$

Step 3 E is commutative

Put

$$A := \{\beta \in [1, \infty[\mid x, y \in E_+ \Longrightarrow \beta(\operatorname{im} xy)^2 \leq (\operatorname{re} xy)^2\} ,$$

$$\gamma := \sup_{\beta \in A} \beta \leq \infty .$$

By Step 2, $1 \in A$. Suppose γ is finite. Then $\gamma \in A$. Take $x, y \in E_+$ and

$$a := \operatorname{re} xy , \quad b := \operatorname{im} xy .$$

Then, we see successively that

$$\gamma b^2 \leq a^2 ,$$

$$a^2 - \gamma b^2 , b^2 \in E_+ \qquad \text{(Theorem 4.2.2.1 c} \Rightarrow \text{a)} ,$$

$$0 \leq 2\operatorname{re} b^2(a^2 - \gamma b^2) = b^2(a^2 - \gamma b^2) + (a^2 - \gamma b^2)b^2 =$$

$$= a^2b^2 + b^2a^2 - 2\gamma b^4 ,$$

$$-a^2b^2 - b^2a^2 \leq -2\gamma b^4$$

by Step 2. Since yxy, ab^2a, a, and $\operatorname{re} a(bab)$ are all positive (Corollary 4.2.2.3 and Step 2) and since

$$x(yxy) = (xy)^2 = (a + ib)^2 = a^2 - b^2 + i(ab + ba) ,$$

we see successively that

$$\gamma(ab + ba)^2 \leq (a^2 - b^2)^2 ,$$

$$\gamma^2 b^4 \leq \gamma ba^2 b \leq \gamma(ab^2a + ba^2b + 2\operatorname{re} a(bab)) =$$

$$= \gamma(ab^2a + ba^2b + abab + baba) = \gamma(ab + ba)^2 \leq (a^2 - b^2)^2 =$$

$$= a^4 + b^4 - a^2b^2 - b^2a^2 \leq a^4 + b^4 - 2\gamma b^4 \text{ (Corollary 4.2.2.3)},$$

$$(\gamma^2 + 2\gamma - 1)b^4 \leq a^4,$$

$$\sqrt{\gamma^2 + 2\gamma - 1}\, b^2 \leq a^2 \text{ (Proposition 4.2.4.8)},$$

$$\sqrt{\gamma^2 + 2\gamma - 1} \in A,$$

$$\sqrt{\gamma^2 + 2\gamma - 1} \leq \gamma,$$

$$1 \leq \gamma \leq \frac{1}{2}$$

which is a contradiction. Hence A cannot be bounded,

$$(\operatorname{im} xy)^2 = 0,$$

$$\|xy - yx\|^2 = \|2i(\operatorname{im} xy)\|^2 = 4\|(\operatorname{im} xy)^2\| = 0,$$

$$xy = yx.$$

By Theorem 4.2.2.9 a), $\operatorname{Re} E$ is commutative. ■

Proposition 4.2.4.12 (0) *Take $x' \in E'_+$. Let A be a nonempty downward directed set of E_+ with*

$$\inf_{x \in A} x'(x) = 0.$$

Let $\mathfrak{F}$ be the lower section filter of A and take $\alpha > 0$. Then

$$\lim_{x, \mathfrak{F}} x'(x^\alpha) = 0.$$

Case 1 $\alpha > \frac{1}{2}$

We may assume that E is unital (Proposition 2.3.4.11) and that 1 is an upper bound for A. Then for $x \in A$,

$$(x'(x^\alpha))^2 = \left(x'\left(x^{\alpha - \frac{1}{2}} x^{\frac{1}{2}}\right)\right)^2 \leq x'(x^{2\alpha - 1})x'(x) \leq x'(1)x'(x)$$

(Proposition 2.3.4.6 c), Corollary 4.2.1.11), so that

$$\lim_{x,\mathfrak{F}} x'(x^\alpha) = 0\,.$$

Case 2 $\alpha \leq 1$

There is some $n \in \mathbb{N}$ with $\alpha^{\frac{1}{n}} > \frac{1}{2}$. Given $k \in \mathbb{N}$, define

$$f_k : E_+ \longrightarrow E_+\,, \quad x \longmapsto x^{\alpha^{\frac{k}{n}}}\,.$$

Then, given any $k \in \mathbb{N}$, $f_k(A)$ is a nonempty downward directed set of E_+ and $f_k(\mathfrak{F})$ is the lower section filter of $f_k(A)$ (Proposition 4.2.4.8). By Case 1,

$$\lim_{x\in\mathfrak{F}} x'(f_1(x)) = 0\,.$$

Since

$$f_{k+1} = f_1 \circ f_k$$

for every $k \in \mathbb{N}$ (Proposition 4.2.4.2 b)), complete induction shows that

$$\lim_{x,\mathfrak{F}} x'(f_k(x)) = 0$$

for all $k \in \mathbb{N}$. For $k = n$,

$$\lim_{x,\mathfrak{F}} x'(x^\alpha) = 0\,. \qquad \blacksquare$$

Proposition 4.2.4.13 (0) *Let E be unital. Take $x, y \in E$, $z \in E_+$, $u \in E(z,1)$, and $\alpha, \beta \in \mathbb{R}_+$ with*

$$x^*x \leq z^\alpha\,, \quad yy^* \leq z^\beta\,, \quad \alpha + \beta > 1\,.$$

Then the sequence $\left(x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} y\right)_{n\in\mathbb{N}}$ converges and

$$\left\| \lim_{n\to\infty} x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} y \right\| \leq \left\| z^{\frac{\alpha+\beta-1}{2}} \right\|\,,$$

$$\lim_{n\to\infty} x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} z^{\frac{1}{2}} = x\,,$$

$$\|xu\| \leq \left\| z^{\frac{\alpha}{2}} u \right\|\,, \quad \|xuy\| \leq \left\| uz^{\frac{\alpha+\beta}{2}} \right\|\,.$$

We have

$$\|xu\|^2 = \|u^*x^*xu\| \leq \|u^*z^\alpha u\| = \|z^{\frac{\alpha}{2}}u\|^2,$$

$$\|xuy\|^2 = \|y^*u^*x^*xuy\| \leq \|y^*u^*z^\alpha uy\| = \|z^{\frac{\alpha}{2}}uy\|^2 =$$

$$= \|z^{\frac{\alpha}{2}}uyy^*u^*z^{\frac{\alpha}{2}}\| \leq \|z^{\frac{\alpha}{2}}uz^\beta u^*z^{\frac{\alpha}{2}}\| = \|z^{\frac{\alpha}{2}}uz^{\frac{\beta}{2}}\|^2 = \|uz^{\frac{\alpha+\beta}{2}}\|^2$$

(Corollary 4.2.2.3, Corollary 4.2.1.18, Corollary 2.3.2.16).

Given $m, n \in \mathbb{N}$, put

$$f_n : \sigma(z) \longrightarrow \mathbb{K}, \quad \gamma \longmapsto \left(\frac{\gamma^{\alpha+\beta}}{\frac{1}{n}+\gamma}\right)^{\frac{1}{2}},$$

$$g_n : \sigma(z) \longrightarrow \mathbb{K}, \quad \gamma \longmapsto \gamma^{\frac{\alpha}{2}}\left(1 - \left(\frac{\gamma}{\frac{1}{n}+\gamma}\right)^{\frac{1}{2}}\right),$$

$$z_n := x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}}y,$$

$$z_{mn} := \left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} - \left(\frac{1}{m}1 + z\right)^{-\frac{1}{2}}.$$

By the above inequalities, Proposition 4.2.4.5 b), and Theorem 4.1.3.1 b),

$$\left\|x - x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}}z^{\frac{1}{2}}\right\| = \left\|x\left(1 - \left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}}z^{\frac{1}{2}}\right)\right\| \leq$$

$$\leq \left\|z^{\frac{\alpha}{2}}\left(1 - \left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}}z^{\frac{1}{2}}\right)\right\| = \|g_n(z)\|,$$

$$\|z_n\| \leq \left\|\left(\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}}z^{\frac{\alpha+\beta}{2}}\right)\right\| = \|f_n(z)\|,$$

$$\|z_n - z_m\| = \|xz_{mn}y\| \leq \left\|z_{mn}z^{\frac{\alpha+\beta}{2}}\right\| = \|f_n(z) - f_m(z)\|$$

for all $m, n \in \mathbb{N}$. By Dini's Theorem, $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge uniformly to

$$\sigma(z) \longrightarrow \mathbb{K}\,, \quad \gamma \longmapsto \gamma^{\frac{\alpha+\beta-1}{2}}\,,$$

and

$$\sigma(z) \longrightarrow \mathbb{K}\,, \quad \gamma \longmapsto 0\,,$$

respectively. It follows that

$$\lim_{n\to\infty} f_n(z) = z^{\frac{\alpha+\beta-1}{2}}\,,$$

$$\lim_{n\to\infty} g_n(z) = 0\,,$$

$$x = \lim_{n\to\infty} x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} z^{\frac{1}{2}}$$

(Theorem 4.1.3.1 b)). Hence $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and

$$\|\lim_{n\to\infty} z_n\| = \lim_{n\to\infty} \|z_n\| \leq \lim_{n\to\infty} \|f_n(z)\| = \left\|z^{\frac{\alpha+\beta-1}{2}}\right\|\,. \qquad \blacksquare$$

Corollary 4.2.4.14 (0) *Take* $x \in E$, $z \in E_+$, *and* $\alpha \in]0, \frac{1}{2}[$ *such that*

$$x^*x \leq z\,.$$

Then there is a $y \in E$ *with*

$$x = yz^\alpha\,, \quad \|y\| \leq \|z^{\frac{1}{2}-\alpha}\|\,.$$

Let $\widetilde{E}$ be the unital C^*–algebra associated to E. Given $n \in \mathbb{N}$, put

$$y_n := x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} z^{\frac{1}{2}-\alpha} \in E$$

for all $n \in \mathbb{N}$. By Proposition 4.2.4.13, $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in E$ with

$$\|y\| \leq \|z^{\frac{1}{2}-\alpha}\|\,.$$

Given $n \in \mathbb{N}$,

$$y_n z^\alpha = x\left(\frac{1}{n}1 + z\right)^{-\frac{1}{2}} z^{\frac{1}{2}}\,,$$

so that

$$yz^\alpha = \lim_{n\to\infty} y_n z^\alpha = x$$

(Proposition 4.2.4.13). $\blacksquare$

Corollary 4.2.4.15 (0) (Effros, 1963) *Let F be a closed ideal of E. Take $x \in E$, $y \in F_+$. If*

$$x^*x \leq y,$$

then $x \in F$.

By Proposition 4.2.4.2 d),

$$y^{\frac{1}{3}} \in F.$$

By Corollary 4.2.4.14, there is a $z \in E$ with

$$x = zy^{\frac{1}{3}} \in F. \qquad \blacksquare$$

Proposition 4.2.4.16 (Pedersen, 1969) *Take $x, y, z \in E$, such that*

$$x^*x \leq y^*y + z^*z.$$

Then there are $u, v \in E$, such that

$$u^*u \leq y^*y, \quad v^*v \leq z^*z, \quad xx^* =: uu^* + vv^*.$$

Let $\widetilde{E}$ be the unital C^*-algebra associated to E. Given $n \in \mathbb{N}$, define

$$w := y^*y + z^*z, \quad w_n := \left(\frac{1}{n}1 + w\right)^{-\frac{1}{2}},$$

$$u_n := xw_ny, \quad v_n := xw_nz$$

and set

$$u := \lim_{n\to\infty} u_n, \quad v := \lim_{n\to\infty} v_n$$

(Theorem 4.2.2.1 c $\Rightarrow$ a, Proposition 4.2.4.13). Then $u, v \in E$. We have that

$$u_n^*u_n = y^*w_nx^*xw_ny \leq y^*w_nww_ny \leq y^*y$$

for every $n \in \mathbb{N}$ (Corollary 4.2.2.3, Corollary 4.2.1.12), so that

$$u^*u = \lim_{n\to\infty} u_n^*u_n \leq y^*y$$

(Theorem 4.2.1.1, Propositions 1.7.1.5 a) and 2.2.1.6). It follows

$$v^*v \leq z^*z.$$

Now

$$0 \leq w_n x^* x w_n \leq w_n w w_n \leq 1$$

(Corollary 4.2.2.3, Theorem 4.2.2.1 c $\Rightarrow$ a, Corollary 4.2.1.12) and

$$1 - w_n w w_n = \frac{1}{n} w_n^2 ,$$

so that for every $n \in \mathbb{N}$,

$$\|xx^* - u_n u_n^* - v_n v_n^*\| = \|x(1 - w_n(yy^* + zz^*)w_n)x^*\| =$$

$$= \|x(1 - w_n w w_n)x^*\| = \frac{1}{n}\|x w_n^2 x^*\| = \frac{1}{n}\|w_n x^* x w_n\| \leq \frac{1}{n}$$

(Corollary 4.2.1.17 b $\Rightarrow$ a). Hence

$$uu^* + vv^* = \lim_{n\to\infty}(u_n u_n^* + v_n v_n^*) = xx^* . \qquad \blacksquare$$

Proposition 4.2.4.17 *Let F be a closed ideal of E and G a closed ideal of F. Then G is an ideal of E.*

Take $(x, y) \in E \times G_+$. Then $y^{\frac{1}{2}} \in G \subset F$, so that $xy^{\frac{1}{2}}, y^{\frac{1}{2}}x \in F$ and

$$xy = (xy^{\frac{1}{2}})y^{\frac{1}{2}} \in G , \quad yx = y^{\frac{1}{2}}(y^{\frac{1}{2}}x) \in G .$$

Assume $\mathbb{K} = \mathbb{C}$. Since every element of G is a linear combination of four elements of G_+ (Theorem 4.2.2.9 a)), it follows that G is an ideal of E.

If $\mathbb{K} = \mathbb{R}$, then $\overset{\circ}{F}$ is a closed ideal of $\overset{\circ}{E}$ and $\overset{\circ}{G}$ a closed ideal of $\overset{\circ}{F}$ (Proposition 2.1.5.11 b)). By the above considerations $\overset{\circ}{G}$ is an ideal of $\overset{\circ}{E}$. Hence G is an ideal of E. $\blacksquare$

Proposition 4.2.4.18 (0) *Given $(E_\iota)_{\iota\in I}$ a family of C^*-algebras, let E be their C^*-direct sum. Suppose that F is a C^*-subalgebra of E such that given $\iota, \kappa \in I$ with $\iota \neq \kappa$ and any $a_\iota \in E_\iota$, there is an $x \in F$ with $x_\iota = a_\iota$ and $x_\kappa = 0$. Then $\operatorname{Re} E \subset F$.*

Take $\iota \in I$ and $a_\iota \in (E_\iota)_+^{\#}$.

Step 1 $\quad \kappa \in I\backslash\{\iota\} \Longrightarrow \exists\, x \in F_+^{\#}, x_\iota = a_\iota, x_\kappa = 0$

By the hypothesis, there is a $y \in F$ such that

$$y_\iota = a_\iota^{\frac{1}{2}} , \quad y_\kappa = 0 .$$

Define

$$f : \mathbb{R}_+ \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \inf\{\alpha, 1\}$$

and

$$x := f(y^*y).$$

Then $x \in F_+^{\#}$,

$$x_\iota = a_\iota, \quad x_\kappa = 0$$

(Corollary 4.2.1.12, Proposition 4.1.3.22).

Step 2 $\quad K \in \mathfrak{P}_f(I\backslash\{\iota\}), K \neq \emptyset \Rightarrow \begin{cases} \exists x \in F_+^{\#}, x_\iota = a_\iota, \\ (\kappa \in K \Rightarrow x_\kappa = 0) \end{cases}$

Define

$$n := \operatorname{Card} K.$$

By Step 1, for each $\kappa \in K$, there is a $y^{(\kappa)} \in F^{\#}$ such that

$$y_\iota^{(\kappa)} = a_\iota^{\frac{1}{n}}, \quad y_\kappa^{(\kappa)} = 0.$$

If we endow K with a total order, then

$$x := \prod_{\kappa \in K} y^{(\kappa)}$$

has the required properties.

Step 3 There is an $x \in F^{\#}$, with $x_\iota = a_\iota$, $(\kappa \in I\backslash\{\iota\} \Rightarrow x_\kappa = 0)$

By the hypothesis, there is a $y \in F$ such that

$$y_\iota = a_\iota^{\frac{1}{2}}.$$

Take $n \in \mathbb{N}$. There is a finite subset K of I containing ι such that

$$\sup_{\kappa \in I\backslash K} \|y_\kappa\| < \frac{1}{n}.$$

By Step 2, there is a $z \in F^{\#}$ such that

$$z_\iota = a_\iota^{\frac{1}{2}},$$

$$\kappa \in K \backslash \{\iota\} \Longrightarrow z_\kappa = 0 .$$

Put

$$x^{(n)} := yz .$$

Then

$$x_\iota^{(n)} = y_\iota z_\iota = a_\iota ,$$

$$\kappa \in K \backslash \{\iota\} \Longrightarrow x_\kappa^{(n)} = y_\kappa z_\kappa = 0 ,$$

$$\kappa \in I \backslash K \Longrightarrow \|x_\kappa^{(n)}\| = \|y_\kappa z_\kappa\| \leq \|y_\kappa\| \, \|z_\kappa\| < \frac{1}{n} .$$

$(x^{(n)})_{n \in \mathbb{N}}$ is obviously a Cauchy sequence in F. Then

$$x := \lim_{n \to \infty} x^{(n)}$$

has the desired properties.

Step 4 $\{x \in \operatorname{Re} E \mid \{\iota \in I \mid x_\iota \neq 0\} \text{ is finite}\} \subset F$

The assertion follows immediately from Step 3 (Theorem 4.2.2.9 a)).

Step 5 $\operatorname{Re} E \subset F$

The assertion follows from Step 4. ■

Example 4.2.4.19 *Take*

$$x := \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix} \in (\mathbb{K}_{2,2})_+ \backslash \{0\}$$

and put

$$\begin{bmatrix} a & b \\ \overline{b} & d \end{bmatrix} := x^{\frac{1}{2}} ,$$

$$c := \sqrt{\alpha + \delta + 2\sqrt{\alpha\delta - |\beta|^2}} \in \mathbb{R}_+$$

(Example 4.2.3.5).

a)
$$\begin{cases} a = \frac{1}{2}(c + \frac{\alpha - \delta}{c}) = \frac{\alpha + \sqrt{\alpha\delta - |\beta|^2}}{c} \\ d = \frac{1}{2}(c + \frac{\delta - \alpha}{c}) = \frac{\delta + \sqrt{\alpha\delta - |\beta|^2}}{c} \\ b = \frac{\beta}{c} \end{cases}$$

b) $ad - |b|^2 = \sqrt{\alpha\delta - |\beta|^2}$

c) $x^{\frac{1}{2}} = \frac{1}{c}(x + \sqrt{\alpha\delta - |\beta|^2}\, 1)$

d) *The following are equivalent:*

d_1) $\alpha\delta = |\beta|^2$,

d_2) $x^{\frac{1}{2}} = \frac{1}{\sqrt{\alpha+\delta}}\, x$

d_3) x *is not proportional to* 1 *and* $x^{\frac{1}{2}}$ *is proportional to* x.

e) $x \in \Pr \mathbb{K}_{2,2} \backslash \{1\}$ *iff* $\alpha\delta = |\beta|^2$ *and* $\alpha + \delta = 1$.

a) By Example 4.2.3.5,

$$\alpha, \delta, \alpha\delta - |\beta|^2, a, d, ad - |b|^2 \in \mathbb{R}_+$$

and $c \neq 0$. We have

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix} = \begin{bmatrix} a^2 + |b|^2 & b(a+d) \\ \overline{b}(a+d) & d^2 + |b|^2 \end{bmatrix}$$

so that

$$\begin{cases} a^2 + |b|^2 = \alpha \\ d^2 + |b|^2 = \delta \\ b(a+d) = \beta \end{cases}$$

Put

$$t := a + d.$$

Then $t > 0$ and

$$\alpha - \delta = a^2 - d^2 = (a-d)(a+d) = t(a-d)$$

so that

$$\begin{cases} a+d=t \\ a-d=\frac{\alpha-\delta}{t} \end{cases}$$

and

$$\begin{cases} a=\frac{1}{2}(t+\frac{\alpha-\delta}{t}) \\ d=\frac{1}{2}(t+\frac{\delta-\alpha}{t}) \end{cases}.$$

It follows

$$\alpha t^2 = a^2t^2 + |b|^2t^2 = \frac{1}{4}(t^4 + 2(\alpha-\delta)t^2 + (\alpha-\delta)^2) + |\beta|^2,$$

$$0 = t^4 - 2(\alpha+\delta)t^2 + (\alpha-\delta)^2 + 4|\beta|^2 =$$

$$= t^4 - 2(\alpha+\delta)t^2 + (\alpha+\delta)^2 - 4\alpha\delta + 4|\beta|^2 =$$

$$= (t^2 - (\alpha+\delta))^2 - 4(\alpha\delta - |\beta|^2),$$

$$t^2 - (\alpha+\delta) = 2\sqrt{\alpha\delta - |\beta|^2},$$

$$t = \sqrt{\alpha+\delta+2\sqrt{\alpha\delta-|\beta|^2}} = c.$$

$$a = \frac{1}{2c}(c^2+\alpha-\delta) = \frac{1}{2c}(\alpha+\delta+2\sqrt{\alpha\delta-|\beta|^2}+\alpha-\delta) =$$

$$= \frac{\alpha+\sqrt{\alpha\delta-|\beta|^2}}{c},$$

$$d = \frac{1}{2c}(c^2+\delta-\alpha) = \frac{1}{2c}(\alpha+\delta+2\sqrt{\alpha\delta-|\beta|^2}+\delta-\alpha) =$$

$$= \frac{\delta+\sqrt{\alpha\delta-|\beta|^2}}{c}.$$

b) By a),

$$ad - |b|^2 = \frac{1}{c^2}\left(\alpha\delta + (\alpha+\delta)\sqrt{\alpha\delta-|\beta|^2} + \alpha\delta - |\beta|^2 - |\beta|^2\right) =$$

$$= \frac{2(\alpha\delta - |\beta|^2 + (\alpha+\delta)\sqrt{\alpha\delta-|\beta|^2}}{c^2} =$$

$$= \frac{c^2\sqrt{\alpha\delta - |\beta|^2}}{c^2} = \sqrt{\alpha\delta - |\beta|^2}\,.$$

c) follows from a).

$d_1 \Rightarrow d_2$ follows from c).

$d_2 \Rightarrow d_3$. Assume x proportional to 1. Then $\alpha = \delta$, $\beta = 0$, $x = \alpha 1$,

$$x^{\frac{1}{2}} = \sqrt{\alpha}1 = \frac{1}{\sqrt{\alpha}}x \neq \frac{1}{\sqrt{\alpha+\delta}}x\,.$$

Hence x is not proportional to 1.

$d_3 \Rightarrow d_1$. There is a $\gamma \in \mathbb{K}$ such that

$$x^{\frac{1}{2}} = \gamma x\,.$$

By c),

$$(\gamma c - 1)x = \sqrt{\alpha\delta - |\beta|^2}1$$

so that

$$\alpha\delta - |\beta|^2 = 0\,.$$

e) If $x \in \Pr \mathbb{K}_{2,2}\backslash\{1\}$ then by Example 4.1.2.19, $\alpha\delta = |\beta|^2$ and $\alpha+\delta = 1$. The reverse implication follows from $d_1 \Rightarrow d_2$ & d_3. ■

Remark. The above assertion still hold for

$$x \in (\mathbb{H}_{2,2})_+\backslash\{0\}\,.$$

Example 4.2.4.20 *Take*

$$x := \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix} \in (\mathbb{K}_{22})_+\backslash\{1\}$$

and put

$$\rho := \sqrt{(\alpha-\delta)^2 + 4|\beta|^2}\,,$$

$$a := \frac{\alpha+\delta+\rho}{2}\,, \quad b := \frac{\alpha+\delta-\rho}{2}\,.$$

Then for every $\gamma > 0$,

$$x^\gamma = \frac{\alpha^\gamma - b^\gamma}{\rho} + \frac{ab^\gamma - ba^\gamma}{\rho}1\,.$$

The assertion follows from Example 4.1.3.19. ■

4.2.5 The Modulus

Definition 4.2.5.1 (0) *Given $x \in E$, define*

$$|x| := (x^*x)^{\frac{1}{2}}$$

for every $x \in E$ (Theorem 4.2.2.1 c $\Rightarrow$ a). $|x|$ is called ***the modulus of*** *x.*

Another candidate for the modulus would be $(xx^*)^{\frac{1}{2}}$. Now

$$(xx^*)^{\frac{1}{2}} = (x^*x)^{\frac{1}{2}}$$

iff x is normal.

Proposition 4.2.5.2 *The map*

$$E \longrightarrow E_+, \quad x \longmapsto |x|$$

is continuous.

This follows from Proposition 4.2.4.2 a). ■

Proposition 4.2.5.3 (0) *If $x \in E$, then $|x|$ belongs to the closed real subalgebra of E generated by x^*x, and x and $|x|$ generate the same closed ideal of E.*

The first assertion follows from Proposition 4.2.4.2 d). Let F and G be the closed ideals of E generated by x and $|x|$, respectively. Then

$$|x|^2 = x^*x \in F, \quad x^*x = |x|^2 \in G.$$

Thus $|x| \in F$ and $x \in G$ by Corollary 4.2.4.15. ■

Proposition 4.2.5.4 (0) *Take $x \in \operatorname{Sn} E$ and*

$$f : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto |\alpha|.$$

Then $|x| = f(x)$ and in the complex case

$$|\operatorname{re} x| + |\operatorname{im} x| \geq |x| = ((\operatorname{re} x)^2 + (\operatorname{im} x)^2)^{\frac{1}{2}} \geq |\operatorname{re} x| \quad \textit{(resp. } |\operatorname{im} x| \textit{)}.$$

Define

$$g : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha .$$

Then

$$f^2 = \overline{g} g ,$$

so that

$$f(x)^2 = f^2(x) = \overline{g}(x) g(x) = x^* x .$$

Since $f(x)$ is positive (Corollary 4.2.1.12),

$$f(x) = |x| .$$

Moreover, in the complex case,

$$|\mathrm{re}\, x| + |\mathrm{im}\, x| \geq |x| = ((\mathrm{re}\, x)^2 + (\mathrm{im}\, x)^2)^{\frac{1}{2}} \geq |\mathrm{re}\, x| \quad (\text{resp. } |\mathrm{im}\, x|)$$

follows from Corollary 2.3.1.24 c), Theorem 4.2.2.1 c $\Rightarrow$ a, and Proposition 4.2.1.12. ■

Remark. If

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then

$$|x| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathrm{re}\, x = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix},$$

and so

$$\mathrm{re}\, x \leq |x|$$

does not hold.

Proposition 4.2.5.5 (0) *Let E, F be C^*-algebras and $u : E \to F$ a homomorphism of involutive algebras. Take $x \in E$.*

a) $u|x| = |ux|$.

b) $x \in \mathrm{Re}\, E \Rightarrow ux^+ = (ux)^+ , ux^- = (ux)^-$.

c) $x \in E_+$, $\alpha \in \mathbb{R}_+ \Rightarrow ux^\alpha = (ux)^\alpha$.

d) $u(E_+) = F_+ \cap u(E)$.

e) *If in addition* u *is surjective then*

$$y' \in F'_+ \Longleftrightarrow u'y' \in E'_+$$

for every $y' \in \operatorname{Re} F'$.

a) Since $u|x| \in F_+$ (Corollary 4.2.1.4),

$$u(|x|)^2 = (u|x|^2) = u(x^*x) = (ux)^*(ux)\,,$$

$$u|x| = |ux|\,.$$

b) Now ux^+, $ux^- \in F_+$ (Corollary 4.2.1.4),

$$(ux^+)(ux^-) = u(x^+x^-) = 0\,, \quad u(x^-)u(x^+) = u(x^-x^+) = 0\,,$$

$$ux = u(x^+ - x^-) = ux^+ - ux^-\,,$$

so that

$$(ux)^+ = ux^+\,, \quad (ux)^- = ux^-\,.$$

c) follows from Corollary 4.1.3.4 (and Definition 4.2.4.1).

d) By Corollary 4.2.1.4,

$$u(E_+) \subset F_+ \cap u(E)\,.$$

Let $y \in F_+ \cap u(E)$ and let $x \in E$ such that $ux = y$. By a),

$$y = |y| = |ux| = u|x| \in u(E_+)\,,$$

$$F_+ \cap u(E) \subset u(E_+)\,.$$

e) Assume $u'y' \in E'_+$. Take $y \in F_+$. By d), there is an $x \in E_+$ with

$$ux = y\,.$$

It follows

$$\langle y, y' \rangle = \langle ux, y' \rangle = \langle x, u'y' \rangle \in \mathbb{R}_+\,,$$

$y' \in F'_+$ (Corollary 4.2.2.10).

The reverse implication was proved in Corollary 4.2.2.11. ■

Remark. A similar result to d), namely

$$u(\Pr E) = \Pr F \cap u(E)$$

does not hold. While the inclusion from left to right is obvious, the reverse inclusion fails, as shown by the example

$$E := \mathcal{C}([0,1]), \quad F := \mathcal{C}(\{0,1\}),$$

$$u : E \longrightarrow F, \quad x \longmapsto x|\{0,1\}.$$

Theorem 4.2.5.6 (0) *Take* $x, y \in E$.

a) $\| |x| \| = \|x\|$.

b) $x \in \operatorname{Re} E \Rightarrow |x| = x^+ + x^- \geq x$, $\begin{cases} x^+ = \frac{1}{2}(|x| + x) \\ x^- = \frac{1}{2}(|x| - x). \end{cases}$

c) $\alpha \in \mathbb{K} \Rightarrow |\alpha x| = |\alpha|\,|x|$.

d) $x \in \operatorname{Re} E$, $|x| \in \Pr E \Rightarrow x^+, x^- \in \Pr E$.

e) $|x+y|^2 \leq 2(|x|^2 + |y|^2)$.

f) $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$.

a) $\| |x| \|^2 = \| |x|^2 \| = \|x^* x\| = \|x\|^2$ (Proposition 4.2.4.2 b)).

b) Since

$$(x^+ + x^-)^2 = (x^+)^2 + (x^-)^2 = (x^+ - x^-)^2 = x^2 = |x|^2$$

(Theorem 4.2.2.9 a)),

$$|x| = x^+ + x^- \geq x^+ - x^- = x$$

(Theorem 4.2.2.9 a), Proposition 4.2.4.8).

c) follows immediately from the definition.

d) By Proposition 4.2.5.4 (and Corollary 4.1.3.5, Proposition 4.1.2.21 a $\Rightarrow$ c),

$$\{|\alpha| \mid \alpha \in \sigma(x)\} = \sigma(|x|) \subset \{0,1\},$$

so that

$$\sigma(x) \subset \{-1,0,1\}.$$

Thus

$$\sigma(x^+) \cup \sigma(x^-) \subset \{0,1\}$$

(Corollary 4.1.3.5, Theorem 4.2.2.9 b)), and so $x^+, x^- \in \Pr E$ (Proposition 4.1.2.21 c $\Rightarrow$ a).

e) By Corollary 4.2.2.4,

$$|x+y|^2 = (x^*+y^*)(x+y) = x^*x + y^*y + x^*y + y^*x \leq$$

$$\leq 2(x^*x + y^*y) = 2(|x|^2 + |y|^2).$$

f) By a) and Corollary 4.2.1.18,

$$\|x\| = \| \, |x| \, \| \leq \| \, |y| \, \| = \|y\|. \qquad \blacksquare$$

Remark. a) For no $\alpha \in \mathbb{R}_+$, a relation of the form

$$|x+y| \leq \alpha(|x| + |y|)$$

holds as it can be seen from the example

$$x := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad y := -\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

with $\theta \in \mathbb{R}$.

b) If we set

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$\operatorname{re} x \not\leq \alpha|x|$$

for all $\alpha \in \mathbb{R}_+$.

Example 4.2.5.7 *Let*

$$x := \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix}$$

be a selfadjoint element of $\mathcal{L}(\mathbb{K}^2)$ *which is neither positive nor negative and put*

$$\rho := \sqrt{(\alpha - \delta)^2 + 4|\beta|^2} \in \mathbb{R}_+ .$$

Then

$$|x| = \frac{1}{\rho} \begin{bmatrix} \alpha^2 - \alpha\delta + 2|\beta|^2 & (\alpha + \delta)\overline{\beta} \\ (\alpha + \delta)\overline{\beta} & \delta^2 - \alpha\delta + 2|\beta|^2 \end{bmatrix} =$$

$$= \frac{\alpha + \delta}{\rho} \begin{bmatrix} \frac{\alpha - \delta}{2} & \beta \\ \overline{\beta} & \frac{\delta - \alpha}{2} \end{bmatrix} + \frac{\rho}{2} 1 .$$

The assertion follows from Theorem 4.2.5.6 b) and Example 4.2.3.7. ■

Proposition 4.2.5.8 *If* E *is a real* C^**-algebra, then*

$$|(0, x)| = (|x|, 0) ,$$

$$(0, x) \geq 0 \Longrightarrow x = 0$$

for every $x \in E$.

We have

$$(0, x)^*(0, x) = (0, -x^*)(0, x) = (x^* x, 0) = (|x|^2, 0) = (|x|, 0)^2 .$$

Since $(|x|, 0)$ is positive,

$$|(0, x)| = (|x|, 0) .$$

Moreover,

$$(0, x) \geq 0$$

implies that

$$(0, x) = |(0, x)| = (|x|, 0) .$$

Hence

$$x = 0 .$$

■

Corollary 4.2.5.9 *Let E be a real C^*-algebra and take $x \in E$. If $x^* = -x$ then $(0,x) \in \operatorname{Re} \overset{\circ}{E}$ and*

$$(0,x)^+ = \frac{1}{2}(|x|,x)\,, \quad (0,x)^- = \frac{1}{2}(|x|,-x)\,.$$

We have

$$(0,x)^{\pm} = \frac{1}{2}(|(0,x)| \pm (0,x)| = \frac{1}{2}((|x|,0) \pm (0,x)) = \frac{1}{2}(|x|,\pm x)$$

(Theorem 4.2.5.6 b), Proposition 4.2.5.8). ■

4.2.6 Ideals and Quotients of C^*–Algebras

Proposition 4.2.6.1 (0) *Take $x \in E$ and let F be the closed left ideal of E generated by x. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $F \cap E(|x|)$ such that*

$$\lim_{n\to\infty} xx_n = x$$

and

$$0 \leq x_n \leq x_{n+1} \leq 1$$

for every $n \in \mathbb{N}$.

Now $|x| \in F \cap E_+$ (Proposition 4.2.5.3). Given $n \in \mathbb{N}$, define

$$f_n : \sigma(|x|) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \inf\{n\alpha, 1\},$$

$$g_n : \sigma(|x|) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha(1 - f_n(\alpha)),$$

$$x_n := f_n(|x|) \in E(|x|) \cap E_+ \cap F$$

(Corollary 4.2.1.12, Corollary 4.1.3.3 a)). Then

$$\|xx_n - x\|^2 = \|(xx_n - x)^*(xx_n - x)\| = \|(x_nx^* - x^*)(xx_n - x)\| =$$

$$= \|x_nx^*xx_n - x_nx^*x - x^*xx_n + x^*x\| = \|x^*x(x_n^2 - 2x_n + 1)\| =$$

$$= \|(|x|(1 - x_n))^2\| = \| |x|(1 - x_n)\|^2 = \|g_n(|x|)\|^2 = \|g_n\|^2 = \frac{1}{(4n)^2}$$

for every $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} xx_n = x.$$

Since

$$0 \leq f_n \leq f_{n+1} \leq 1$$

it follows

$$0 \leq x_n \leq x_{n+1} \leq 1$$

for every $n \in \mathbb{N}$ (Corollary 4.2.1.12). ■

Corollary 4.2.6.2 (0) (Segal, 1949) *Every closed ideal of E is involutive.*

Let F be a closed ideal of E and take $x \in F$. By Proposition 4.2.6.1, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in F_+ with

$$x = \lim_{n\to\infty} xx_n .$$

Thus

$$x^* = \lim_{n\to\infty} x_n x^* \in F ,$$

since $x_n x^* \in F$ for every $n \in \mathbb{N}$. Hence F is involutive. ■

Proposition 4.2.6.3 (Størmer, 1967) *If F and G are closed ideals of E, then*

$$(F+G)\cap E_+ = F\cap E_+ + G\cap E_+ .$$

Take $x \in (F+G)\cap E_+$. There is a $(y,z) \in F\times G$ with

$$x = y + z .$$

We may assume y, z to be selfadjoint (Corollary 4.2.6.2). Then

$$x \le |y| + |z|$$

(Proposition 4.2.5.6 b)). By Proposition 4.2.4.16 (and Proposition 4.2.4.2 b)), there are $u, v \in E$ such that

$$u^*u \le |y| , \quad v^*v \le |z| , \quad x = uu^* + vv^* .$$

Then $|y| \in F$, $|z| \in G$ (Proposition 4.2.5.3), so that $u^*u \in F$, $v^*v \in G$ (Corollary 4.2.4.15). Hence

$$(uu^*)^2 = uu^*uu^* \in F ,$$

$$uu^* = ((uu^*)^2)^{\frac{1}{2}} \in F$$

(Proposition 4.2.4.2 d)),

$$vv^* \in G ,$$

$$x \in F\cap E_+ + G\cap E_+$$

(Theorem 4.2.2.1 c $\Rightarrow$ a),

$$(F+G)\cap E_+ \subset F\cap E_+ + G\cap E_+ .$$

The reverse inclusion is trivial. ■

Proposition 4.2.6.4 (0) *If F is a closed ideal of E, then*

$$\|X\| = \inf\left\{\|x - xy\| \,\middle|\, y \in F_+^\#\right\}$$

for every $X \in E/F$ and $x \in X$.

Define

$$\alpha := \inf\left\{\|x - xy\| \,\middle|\, y \in F_+^\#\right\} .$$

Take $z \in X$. Then $x - z \in F$, so, by Proposition 4.2.6.1, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in F such that

$$0 \leq x_n \leq 1$$

for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} (x - z)x_n = x - z .$$

Take $n \in \mathbb{N}$. Then x_n, $1 - x_n \in F_+^\#$ (Corollary 4.2.1.17 b $\Rightarrow$ a & c). Hence

$$\alpha \leq \|x - xx_n\| = \|(x - z) - (x - z)x_n + z(1 - x_n)\| \leq$$

$$\leq \|(x - z) - (x - z)x_n\| + \|z\|\,\|1 - x_n\| \leq \|(x - z) - (x - z)x_n\| + \|z\| ,$$

$$\alpha \leq \|z\| ,$$

$$\alpha \leq \|X\| .$$

The reverse inequality is trivial. ■

Theorem 4.2.6.5 (0) (Segal, Kaplanski, 1949) *If E is a closed ideal of E, then E/F is a C^*-algebra (Corollary 4.2.6.2).*

By Proposition 2.3.2.10 (and Theorem 1.2.4.2 e)), E/F is an involutive Banach algebra and the quotient map $E \to E/F$ is a homomorphism of involutive algebras.

Take $X \in E$, $x \in X$, and $y \in F_+^\#$. We have

$$x^*x \in X^*X , \quad x(1 - y) \in X , \quad \|1 - y\| \leq 1$$

(Corollary 4.2.1.17 a $\Rightarrow$ c). Thus, by Proposition 4.2.6.4,

$$\|X\|^2 \leq \|x(1-y)\|^2 = \|(1-y)x^*x(1-y)\| \leq$$

$$\leq \|1-y\|\,\|x^*x(1-y)\| \leq \|x^*x(1-y)\|\,,$$

$$\|X\|^2 \leq \inf\left\{\|x^*x(1-y)\| \mid y \in F_+^{\#}\right\} = \|X^*X\|\,.$$

Therefore E/F is a C^*–algebra in the complex case (Proposition 4.1.1.3). If $\mathbb{K} = \mathbb{R}$, then the claim follows from the above considerations and Proposition 2.3.1.42. ■

Theorem 4.2.6.6 (0) (Gelfand–Naimark, 1948) *Let E, F be C^*–algebras and $u : E \to F$ a homomorphism of involutive algebras. Then $\operatorname{Im} u$ is a C^*–subalgebra of F and the associated algebraic isomorphism*

$$E/\operatorname{Ker} u \longrightarrow \operatorname{Im} u$$

of u is an isometry of C^*–algebras (Theorem 4.2.6.5).

First assume that u is injective. By Corollary 4.1.3.4, $\|ux\| = \|x\|$ for every $x \in E$. By Proposition 1.2.1.18 c), $\operatorname{Im} u$ is closed. Hence $\operatorname{Im} u$ is a C^*–subalgebra of F (Proposition 2.3.1.16) and the map

$$E \longrightarrow \operatorname{Im} u\,, \quad x \longmapsto ux$$

is an isometry of C^*–algebras.

Now let u be arbitrary. Then $\operatorname{Ker} u$ is a closed ideal of E (Proposition 2.1.1.10). Let

$$v : E/\operatorname{Ker} u \longrightarrow F$$

be the factorization of u through $E/\operatorname{Ker} u$. $E/\operatorname{Ker} u$ is a C^*–algebra (Theorem 4.2.6.5) and v is injective. It is easy to see that v is an involutive algebra homomorphism. By the above proof, $\operatorname{Im} u = \operatorname{Im} v$ is a C^*–subalgebra of F and the map

$$E/\operatorname{Ker} u \longrightarrow \operatorname{Im} u\,, \quad A \longmapsto vA$$

is an isometry of C^*–algebras. ■

Corollary 4.2.6.7 (7) *If F is a closed ideal of E and G is a C^*–subalgebra of E then $F+G$ is the C^*–subalgebra of E generated by $F \cup G$ and the C^*–algebras $(F+G)/F$ and $G/F \cap G$ (Theorem 4.2.6.5) are canonically isometric. If in addition $F \cap G = \{0\}$, then*

$$\|y\| \leq \|x+y\|$$

for all $(x, y) \in F \times G$.

It is obvious that $F + G$ is the involutive subalgebra of F generated by $F \cup G$ (Corollary 4.2.6.2). We show that $F + G$ is closed. Let $q : E \to E/F$ be the quotient map. By Theorem 4.2.6.6, $q(G)$ is closed and the associated algebraic isomorphism

$$G/(F \cap G) \longrightarrow q(G) = (F + G)/F$$

of $q|G$ is an isometry. Since

$$F + G = \overset{-1}{q}(q(G)),$$

$F + G$ is closed.

The last assertion follows from Corollary 4.1.1.20, since the map

$$F + G \longrightarrow G, \quad x + y \longmapsto y$$

is an involutive algebra homomorphism. ■

Corollary 4.2.6.8 *Let T be a compact space, $\mathcal{F}$ a closed ideal of $\mathcal{C}(T)$ and*

$$S := \bigcap_{x \in \mathcal{F}} \overset{-1}{x}(0).$$

Then the factorization

$$\mathcal{C}(T)/\mathcal{F} \longrightarrow \mathcal{C}(S)$$

of the map

$$\mathcal{C}(T) \longrightarrow \mathcal{C}(S), \quad x \longmapsto x|S$$

is an isometry of C^*-algebras (Theorem 4.2.6.5).

Now

$$\mathcal{F} = \{x \in \mathcal{C}(T) \mid x|S = 0\}$$

(Corollary 1.3.5.18) and the map

$$\mathcal{C}(T) \longrightarrow \mathcal{C}(S), \quad x \longmapsto x|S$$

is a surjective homomorphism of involutive algebras. The assertion therefore follows from Theorem 4.2.6.6. ■

Theorem 4.2.6.9 (0) *Let x be an invertible element of E. Then $|x|$ is invertible, $x|x|^{-1}$ is unitary, and there is a uniquely determined $(y,z) \in \mathrm{Un}\, E \times E_+$ such that*

$$x = yz\,.$$

(y,z) is called ***the polar representation of*** *x. Furthermore*

$$y = x|x|^{-1} \in E(x) = E(x,1)\,, \quad z = |x| \in E(x)\,.$$

x^*, x^*x, and $|x|$ are invertible (Proposition 2.3.1.14, Corollary 2.1.2.6, Proposition 4.2.4.5 a)). Since

$$(x|x|^{-1})(x|x|^{-1})^* = x|x|^{-2}x^* = x(x^*x)^{-1}x^* = xx^{-1}(x^*)^{-1}x^* = 1\,,$$

$$(x|x|^{-1})^*(x|x|^{-1}) = |x|^{-1}x^*x|x|^{-1} = |x|^{-1}|x|^2|x|^{-1} = 1\,,$$

$x|x|^{-1}$ is unitary. This establishes the existence of a polar representation. We now prove its uniqueness.

$$z^2 = zy^*yz = (yz)^*(yz) = x^*x = |x|^2$$

and so

$$z = |x|\,, \quad y = x|x|^{-1}\,.$$

By Theorem 4.2.5.3, $|x| \in E(x)$. Thus $|x|^{-1} \in E(x)$ (Proposition 4.2.4.5 b)) and $x|x|^{-1} \in E(x) = E(x,1)$. ■

Corollary 4.2.6.10 *The map*

$$\mathrm{Un}\, E \times \{z \in E_+ \mid z \text{ is invertible}\} \longrightarrow \{x \in E \mid x \text{ is invertible}\}\,,$$

$$(y,z) \longmapsto yz$$

is a homeomorphism.

The map is bijective (Theorem 4.2.6.9) and continuous. Since the map

$$\{x \in E \mid x \text{ is invertible}\} \longrightarrow E_+\,, \quad x \longmapsto |x|$$

is continuous (Proposition 4.2.5.2), the map

$$\{x \in E \mid x \text{ is invertible}\} \longrightarrow E\,, \quad x \longmapsto x|x|^{-1}$$

is also continuous (Theorem 2.2.4.14). Hence the inverse of the above map is continuous (Theorem 4.2.6.9). ■

Corollary 4.2.6.11 *Take $x, y \in \operatorname{Sn} E$. Let z be an invertible element of E and (u, v) the polar representation of z. If*

$$y = zxz^{-1},$$

then

$$y = uxu^{-1}.$$

We have, successively, that

$$yz = zx,$$

$$y^*z = zx^*$$

(Theorem 4.1.4.1),

$$xz^* = z^*y,$$

$$xv^2 = xz^*z = z^*yz = z^*zx = v^2x,$$

$$xv = vx$$

(Corollary 4.2.4.2 b)),

$$y = uvxv^{-1}u^{-1} = uxvv^{-1}u^{-1} = uxu^{-1}. \qquad \blacksquare$$

Proposition 4.2.6.12 *Take $x, y \in E$ with y unitary and*

$$x = y|x|.$$

Then x is normal iff

$$y|x| = |x|y.$$

If x is normal, then

$$|x|^2y = x^*xy = xx^*y = y|x|^2y^*y = y|x|^2$$

and so

$$|x|y = y|x|$$

(Proposition 4.2.4.2 b)).

From

$$y|x| = |x|y$$

it follows that

$$xx^* = y|x|^2y^* = |x|^2yy^* = |x|^2 = x^*x\,,$$

i.e. x is normal. ∎

Corollary 4.2.6.13 *Let x be an invertible element of E and let (y,z) be its polar representation.*

a) *x is normal iff $yz = zy$.*

b) *x is selfadjoint iff $yz = zy$ and y is selfadjoint. In this case,*

$$x^+ = y^+z = y^+x^+\,, \quad x^- = y^-z = y^-x^-\,.$$

a) follows from Proposition 4.2.6.12.

b) Assume x to be selfadjoint. By a),

$$x = x^* = y^*z$$

and so (y^*, z) is the polar representation of x. Thus $y^* = y$ and the C^*-subalgebra of E generated by $\{y,z\}$ is a Gelfand C^*-algebra (Corollary 4.1.2.3). By Corollary 4.1.2.5,

$$x^+ = y^+z\,, \quad x^- = y^-z\,.$$

Since

$$y^+x^- = y^+y^-z = 0\,, \quad y^-x^+ = y^-y^+z = 0\,,$$

we see that by Theorem 4.2.5.6 b),

$$y^+z = y^+(x^+ + x^-) = y^+x^+\,,$$

$$y^-z = y^-(x^+ + x^-) = y^-x^-\,.$$

If $yz = zy$ and y is selfadjoint, then

$$x^* = zy^* = zy = yz = x\,.$$ ∎

Proposition 4.2.6.14 *Let E, F be unital C^*- algebras, $u : E \to F$ a homomorphism of involutive unital algebras, and x an invertible element of E.*

a) $x \in E_+$, $\alpha \in \mathbb{R} \Rightarrow u \log x = \log(ux)$, $ux^\alpha = (ux)^\alpha$.

b) *If (y, z) is the polar representation of x, then (uy, uz) is the polar representation of ux.*

a) follows from Corollary 4.1.3.4 (and Definition 4.2.4.4).

b) uy is unitary (Proposition 2.3.1.16), $uz \in F_+$ (Corollary 4.2.1.4), and

$$(uy)(uz) = u(yz) = ux\,,$$

i.e. (uy, uz) is the polar representation of ux. ■

Proposition 4.2.6.15 *Let E, F be C^*-algebras and $u : E \to F$ a surjective involutive algebra homomorphism. Take $x \in E_+$ and $y \in F$. If*

$$y^*y \leq ux$$

then there is a $z \in \overset{-1}{u}(y)$ such that

$$z^*z \leq x\,.$$

We may assume that E, F, u are unital (Theorem 4.1.1.13, Proposition 2.1.1.7).

Take $a \in \overset{-1}{u}(y)$. Given $n \in \mathbb{N}$, define

$$b := (a^*a - x)^+\,, \quad c := b + x\,,$$

$$c_n := \left(\frac{1}{n}1 + c\right)^{-\frac{1}{2}}\,, \quad z_n := ac_nx^{\frac{1}{2}}\,.$$

Then

$$a^*a \leq b + x = c$$

and hence $(z_n)_{n\in\mathbb{N}}$ converges (Proposition 4.2.4.13). Put

$$z := \lim_{n\to\infty} z_n\,.$$

Since

$$z_n^*z_n = x^{\frac{1}{2}}c_na^*ac_nx^{\frac{1}{2}} \leq x^{\frac{1}{2}}c_ncc_nx^{\frac{1}{2}} \leq x^{\frac{1}{2}}1x^{\frac{1}{2}} = x$$

for every $n \in \mathbb{N}$ (Corollary 4.2.2.3, Corollary 4.2.1.12), we see that

$$z^*z = \lim_{n\to\infty} z_n^* z_n \leq x$$

(Proposition 1.7.1.5). For $n \in \mathbb{N}$

$$ub = (u(a^*a - x))^+ = (y^*y - ux)^+ = 0\,,$$

$$uc_n = \left(\frac{1}{n}1 + uc\right)^{-\frac{1}{2}} = \left(\frac{1}{n}1 + ux\right)^{-\frac{1}{2}}$$

$$uz_n = (ua)\left(\frac{1}{n}1 + ux\right)^{-\frac{1}{2}} (ux)^{\frac{1}{2}}$$

(Proposition 4.2.5.5 b),c), Proposition 4.2.6.14 a)), so that

$$uz = \lim_{n\to\infty} uz_n = ua = y$$

(Proposition 4.2.4.5 a)). ■

Corollary 4.2.6.16 *Let E, F be C^*-algebras, $u : E \to F$ a surjective involutive algebra homomorphism, and $(y_n)_{n\in\mathbb{N}}$ a decreasing sequence in F_+. Then there is a decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in E_+ such that $ux_n = y_n$ for every $n \in \mathbb{N}$.*

We construct the sequence $(x_n)_{n\in\mathbb{N}}$ recursively. By Proposition 4.2.5.5 d), there is an $x_1 \in E_+$ with $ux_1 = y_1$. Take $n \in \mathbb{N}$ and assume the sequence is constructed up to n. By Proposition 4.2.6.15, there is a $z \in \overline{u}^{-1}\left(y_{n+1}^{\frac{1}{2}}\right)$ such that $z^*z \leq x_n$. Put $x_{n+1} := z^*z$. Then $x_{n+1} \in E_+$, $x_{n+1} \leq x_n$, and

$$ux_{n+1} = (uz)^*(uz) = \left(y_{n+1}^{\frac{1}{2}}\right)^* y_{n+1}^{\frac{1}{2}} = y_{n+1}\,.$$ ■

Proposition 4.2.6.17 (0) *Let F be an E-module. If $(x, a) \in E \times F$, then*

$$\|xa\| = \| \, |x|a\| \,, \quad \|ax\| = \|a|x| \, \| \,.$$

We may assume that $\|x\| = 1$. Take $n \in \mathbb{N}$. By taking

$$z := |x|^2\,, \quad \alpha := \frac{n-1}{2n}$$

in Corollary 4.2.4.14, we obtain an element $y_n \in E$ such that

$$x = y_n |x|^{\frac{n-1}{n}}, \quad \|y_n\| \leq \|x\|^{\frac{1}{n}}$$

(Proposition 4.2.4.2 b)). Put

$$z_n := \left(\frac{1}{n}1 + |x|\right)^{-1} |x|^{\frac{1}{n}} x^* \in E,$$

$$f_n := \sigma(|x|) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \frac{\alpha^{\frac{2n+1}{n}}}{\frac{1}{n} + \alpha}.$$

Then

$$z_n x = \left(\frac{1}{n}1 + |x|\right)^{-1} |x|^{\frac{2n+1}{n}},$$

$$\|z_n\| \leq \left\| \left(\frac{1}{n}1 + |x|\right)^{-1} |x|^{\frac{1}{n}} \right\| \|x^*\| \leq \|x\|^{\frac{1}{n}}$$

(Corollary 4.2.1.7),

$$\|xa\| = \left\| y_n |x|^{\frac{n-1}{n}} a \right\| \leq \|y_n\| \left\| |x|^{\frac{n-1}{n}} a \right\| \leq \|x\|^{\frac{1}{n}} \left\| |x|^{\frac{n-1}{n}} a \right\|,$$

$$\left\| \left(\frac{1}{n}1 + |x|\right)^{-1} |x|^{\frac{2n+1}{n}} a \right\| = \|z_n x a\| \leq \|z_n\| \|xa\| \leq \|x\|^{\frac{1}{n}} \|xa\|.$$

Since $(f_n)_{n\in\mathbb{N}}$ is increasing and

$$\lim_{n\to\infty} f_n(\alpha) = \alpha$$

for every $\alpha \in \sigma(|x|)$, it follows, by Dini's Theorem, that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function

$$\sigma(|x|) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha.$$

Hence

$$\lim_{n\to\infty} \left(\frac{1}{n}1 + |x|\right)^{-1} |x|^{\frac{2n+1}{n}} = \lim_{n\to\infty} f_n(|x|) = |x|.$$

We deduce that

$$\|xa\| \leq \lim_{n\to\infty} \|x\|^{\frac{1}{n}} \left\| |x|^{\frac{n-1}{n}} a \right\| = \| |x|a\| =$$

$$= \lim_{n\to\infty} \left\| \left(\frac{1}{n}1 + |x|\right)^{-1} |x|^{\frac{2n+1}{n}} a \right\| \leq \lim \|x\|^{\frac{1}{n}} \|xa\| = \|xa\|,$$

$$\|xa\| = \| |x|a\|. \qquad \blacksquare$$

Proposition 4.2.6.18 *Let $\mathfrak{F}$ be a set of closed ideals of E such that*

$$F \cap G = \{0\}$$

whenever $F, G \in \mathfrak{F}$ are distinct and let H be the C^*-direct sum of $(F)_{F\in\mathfrak{F}}$ (Corollary 4.2.6.2). Define

$$H_0 := \{x \in H \mid \{F \in \mathfrak{F} \mid x_F \neq 0\} \text{ is finite}\},$$

and

$$u_0 : H_0 \longrightarrow E, \quad x \longmapsto \sum_{F\in\mathfrak{F}} x_F .$$

Then there is a unique involutive algebra homomorphism $u : H \to E$ extending u_0. u preserves the norm. If the vector subspace of E generated by $\bigcup_{F\in\mathfrak{F}} F$ is dense in E, then u is an isometry of C^*-algebras.

Since H_0 is dense in H, uniqueness follows from Corollary 4.1.1.20.

Let F, G be distinct elements of $\mathfrak{F}$ and take $(x, y) \in F \times G$. Then

$$xy \in F \cap G,$$

so that $xy = 0$.

Let $\mathfrak{G}$ be a finite subset of $\mathfrak{F}$ and take $x \in \prod_{F\in\mathfrak{G}} F$ such that

$$\sum_{F\in\mathfrak{G}} x_F = 0 .$$

Take $G \in \mathfrak{G}$. By Proposition 4.2.6.1, there is a sequence $(y_n)_{n\in\mathbb{N}}$ in G such that

$$x_G = \lim_{n\to\infty} x_G y_n .$$

By the above considerations,

$$x_G y_n = \sum_{F\in\mathfrak{G}} x_F y_n = 0$$

for every $n \in \mathbb{N}$. Thus $x_G = 0$. Since G is arbitrary, it follows that $x = 0$. Hence the restriction $u_{\mathfrak{G}}$ of u_0 to

$$H_{\mathfrak{G}} := \{x \in H_0 \mid \{F \in \mathfrak{F} \mid x_F \neq 0\} \subset \mathfrak{G}\}$$

is injective. Since $H_{\mathfrak{G}}$ is a C^*-algebra and since $u_{\mathfrak{G}}$ is an involutive algebra homomorphism, it follows that $u_{\mathfrak{G}}$ preserves the norm (Theorem 4.2.6.6). Since $\mathfrak{G}$ is arbitrary, u_0 preserves the norm. Hence, we may extend u_0 to an involutive algebra homomorphism $H \to E$ which preserves the norm.

The last assertion now follows. ■

4.2.7 The Ordered Set of Orthogonal Projections

Proposition 4.2.7.1 (0) *Take $x \in E$ and $p \in \Pr E$. Then the following are equivalent:*

a) $|x| \leq p\,, \quad |x^*| \leq p\,.$

b) $px = xp = x \in E^\#\,.$

c) $pxp = x \in E^\#\,.$

If x is selfadjoint then the above are also equivalent to the following:

d) $-p \leq x \leq p\,.$

e) $px = x \in E^\#\,.$

f) $xp = x \in E^\#\,.$

a $\Rightarrow$ b. Since

$$\|x\| = \| \, |x| \, \| \leq \|p\| \leq 1$$

(Theorem 4.2.5.6 a), Corollary 4.2.1.18), $x \in E^\#$ and

$$0 \leq x^*x = |x|^2 \leq |x| \leq p$$

(Corollary 4.2.1.12). We deduce successively that

$$0 \leq (1-p)x^*x(1-p) \leq (1-p)p(1-p) = 0 \text{ (Corollary 4.2.2.3)}\,,$$

$$\|x(1-p)\|^2 = \|(1-p)x^*x(1-p)\| = 0\,,$$

$$x(1-p) = 0\,,$$

$$x = xp\,,$$

$$x^* = x^*p\,,$$

$$x = px\,.$$

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. We have, successively, that

$$\|x^*x\| = \|x\|^2 \leq 1\,,$$

$$0 \leq x^*x \leq 1 \ \text{(Corollary 4.2.1.17 a} \Rightarrow \text{b)}\,,$$

$$(pxp)^*(pxp) = px^*pxp \leq px^*xp \leq p1p = p \ \text{(Corollary 4.2.2.3)}\,,$$

$$|x| = |pxp| = ((pxp)^*(pxp))^{\frac{1}{2}} \leq p \ \text{(Proposition 4.2.4.8)}\,.$$

Since

$$px^*p = x^* \in E^{\#}\,,$$

it follows from the above relation that

$$|x^*| \leq p\,.$$

a $\Rightarrow$ d follows from Theorem 4.2.5.6 b),c).

d $\Rightarrow$ c. By the implication a $\Rightarrow$ c above, we have successively that

$$0 \leq x + p \leq 2p\,,$$

$$0 \leq \frac{1}{2}(x+p) \leq p\,,$$

$$p\left(\frac{1}{2}(x+p)\right)p = \frac{1}{2}(x+p)\,,$$

$$pxp + p = x + p\,,$$

$$pxp = x\,.$$

Now

$$\|x\| \leq \|p\| \leq 1$$

(Corollary 4.2.1.18), i.e. $x \in E^{\#}$.

b $\Leftrightarrow$ e $\Leftrightarrow$ f is trivial. ■

Corollary 4.2.7.2 (0) *Take* $p \in \Pr E$ *and* $x \in E_+$. *If*

$$x \leq p\,,$$

then

$$\alpha \in]0,1] \Longrightarrow x \leq x^{\alpha} \leq p\,,$$

$$\alpha \in [1,\infty[\Longrightarrow x^{\alpha} \leq x \leq p\,.$$

By Proposition 4.2.7.1 a $\Rightarrow$ b,

$$px = xp = x\,.$$

Hence the C^*-subalgebra F of E generated by $\{x, p\}$ is a Gelfand C^*-algebra (Corollary 4.1.2.3) and the assertions follow by the Gelfand transform on F. ■

Corollary 4.2.7.3 (0) *Given $x \in E_+^{\#}$ and $p \in \operatorname{Pr} E$, the following are equivalent:*

a) $p \leq x$.

b) $px = p$.

c) $xp = p$.

d) $px = xp = p$.

e) $pxp = p$.

f) $xpx = p$.

In particular, every strictly decreasing sequence in $\operatorname{Pr} E$ is linearly independent.

By Corollary 4.2.1.17 a $\Rightarrow$ b & c, $x \leq 1$ and $1 - x \in E_+^{\#}$.
a $\Rightarrow$ b.

$$0 \leq 1 - x \leq 1 - p\,,$$

$$1 - x = (1 - p)(1 - x) = 1 - x - p + px$$

(Proposition 4.2.7.1 d $\Rightarrow$ e), i.e.

$$px = p\,.$$

b $\Rightarrow$ c $\Rightarrow$ d $\Rightarrow$ e and d $\Rightarrow$ f are trivial.
e $\Rightarrow$ b. We have, successively, that

$$x \leq x^{\frac{1}{2}} \leq 1$$

(Corollary 4.2.7.2),

$$p = pxp \leq px^{\frac{1}{2}}p \leq p1p = p$$

(Corollary 4.2.2.3),

$$px^{\frac{1}{2}}p = p\,,$$

$$\left\|px^{\frac{1}{2}} - p\right\|^2 = \left\|\left(px^{\frac{1}{2}} - p\right)\left(x^{\frac{1}{2}}p - p\right)\right\| = \left\|pxp - px^{\frac{1}{2}}p - px^{\frac{1}{2}}p + p\right\| = 0\,,$$

$$px^{\frac{1}{2}} = p\,,$$

$$px = px^{\frac{1}{2}}x^{\frac{1}{2}} = px^{\frac{1}{2}} = p\,.$$

f $\Rightarrow$ a.

$$p = xpx \leq x1x = x^2 \leq x$$

(Corollary 4.2.7.2). ■

Corollary 4.2.7.4 (0) *Given $p, q \in \Pr E$, the following are equivalent:*

a) $pq = qp$.

b) pq *is the infimum of* p *and* q *in* $E_+^{\#}$.

c) $p + q - pq$ *is the supremum of* p *and* q *in* $E_+^{\#}$.

d) pq *and* $p + q - pq$ *belong to* $\Pr E$.

e) $pq \in \operatorname{Re} E$.

a $\Rightarrow$ b & c & d. The C^*–subalgebra F of E generated by p and q is a Gelfand C^*–algebra (Corollary 4.1.2.3). Identifying F with $\mathcal{C}_0(\sigma(F))$ by means of the Gelfand transform, we see that pq (resp. $p + q - pq$) belongs to $\Pr E$ and is a lower (upper) bound for p and q. Let x be a lower (upper) bound for p and q in $E_+^{\#}$. Then

$$px = x\,, \quad qx = x \quad (px = p\,, \quad qx = q)$$

by Proposition 4.2.7.1 d $\Rightarrow$ e (by Corollary 4.2.7.3 a $\Rightarrow$ b). Hence

$$pqx = x$$

$$((p + q - pq)x = px + qx - pqx = p + q - pq)$$

and

$$x \leq pq \quad (p + q - pq \leq x)$$

by Proposition 4.2.7.1 e $\Rightarrow$ d (by Corollary 4.2.7.3 b $\Rightarrow$ a). Thus pq (resp. $p + q - pq$) is the infimum (supremum) of p and q in $E_+^{\#}$.

b $\Rightarrow$ e, c $\Rightarrow$ e, and d $\Rightarrow$ e are trivial.

e $\Rightarrow$ a. We have

$$qp = (pq)^* = pq\,.$$

■

Corollary 4.2.7.5 (0) *Take* $p, q \in \Pr E$. *If*

$$pq = 0\,,$$

then the infimum of p *and* q *in* E_+ *is* 0, $p + q$ *is an orthogonal projection and it is the supremum of* p *and* q *in* $E_+^{\#}$.

The assertion follows immediately from Corollary 4.2.7.4 a $\Rightarrow$ b & c & d.

■

Remark. $p + q$ is not necessarily the supremum of p and q in E_+, as the following example shows. Put

$$E := \mathcal{L}(\mathbb{K}^2)\,, \quad p := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\,, \quad q := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\,.$$

Then

$$p + q = 1\,,$$

but the matrix

$$\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

is larger than p and q and is not larger than 1. p and q have no supremum in E_+, since such a supremum would be smaller than 1, i.e. it would belong to $E_+^{\#}$ and this contradicts the above considerations.

Corollary 4.2.7.6 (0) *Given* $p, q \in \Pr E$, *the following are equivalent:*

a) $p \leq q$.

b) $pq = qp = p$.

c) $pq = p$.

d) $qp = p$.

e) $pqp = p$.

f) $qpq = p$.

g) $q - p \in \Pr E$.

h) $\alpha p \leq q$ *for some* $\alpha > 0$.

i) $pq = \alpha p + \beta q$ *for some* $\alpha \in \mathbb{K}\backslash\{0\}$ *and* $\beta \in \mathbb{K}$.

In particular, every strictly decreasing sequence in $\Pr E$ *is linearly independent.*

By Corollary 4.2.7.3, the assertions a) to f) are equivalent.

b $\Rightarrow$ g. Since

$$(q-p)^2 = q^2 - pq - qp + p^2 = q - p - p + p = q - p,$$

$q - p \in \Pr E$.

g $\Rightarrow$ a $\Rightarrow$ h and c $\Rightarrow$ i are trivial.

h $\Rightarrow$ c. By Proposition 4.2.7.1 d $\Rightarrow$ f,

$$\alpha pq = \alpha p$$

and so

$$pq = p.$$

i $\Rightarrow$ c. Since

$$pq = p^2 q = \alpha p + \beta pq,$$

$$(1-\beta)pq = \alpha p.$$

Since $\alpha \neq 0$, it follows that $\beta \neq 1$. Thus

$$\frac{\alpha}{1-\beta} p = pq = pq^2 = \frac{\alpha}{1-\beta} pq = \left(\frac{\alpha}{1-\beta}\right)^2 p,$$

$$p = \frac{\alpha}{1-\beta} p .$$

Hence either $p = 0$ or $\frac{\alpha}{1-\beta} = 1$ and in both cases

$$pq = p .$$

Now we prove the last assertion. Take $(p_n)_{n\in\mathbb{N}}$ a strictly decreasing sequence in $\operatorname{Pr} E$ and $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{K}^{(\mathbb{N})}$ such that

$$\sum_{n\in\mathbb{N}} \alpha_n p_n = 0 .$$

By a $\Rightarrow$ b, for every $m \in \mathbb{N}$,

$$0 = \sum_{n\in\mathbb{N}} \alpha_n p_n (p_m - p_{m+1}) = \sum_{n\in\mathbb{N}} \alpha_n p_n p_m - \sum_{n\in\mathbb{N}} \alpha_n p_n p_{m+1} =$$

$$= \sum_{n=1}^{m} \alpha_n p_m + \sum_{n=m+1}^{\infty} \alpha_n p_n - \sum_{n=1}^{m} \alpha_n p_{m+1} + \sum_{n=m+1}^{\infty} \alpha_n p_n =$$

$$= \left(\sum_{n=1}^{m} \alpha_n \right) (p_n - p_{m+1}) ,$$

$$\sum_{n=1}^{m} \alpha_n = 0 .$$

Hence $\alpha_n = 0$ for every $n \in \mathbb{N}$ and $(p_n)_{n\in\mathbb{N}}$ is linearly independent. ■

Corollary 4.2.7.7 (0) *Take* $p, q, r \in \operatorname{Pr} E$ *with*

$$p \leq q \leq r .$$

Then $p + (r - q)$ *is an orthogonal projection and it is the supremum of* p *and* $r - q$ *in* $E_+^{\#}$.

By Corollary 4.2.7.6 a $\Rightarrow$ c & g, $r - q \in \operatorname{Pr} E$ and

$$p(r - q) = pr - pq = p - p = 0 .$$

The assertion now follows from Corollary 4.2.7.5. ■

Proposition 4.2.7.8 (0) *Take* $p \in \Pr E$. *Let* F *be a* C^**-subalgebra of* E *such that*

$$(pF) \cup (Fp) \subset F .$$

Put

$$A := \{x \in F \mid 0 \leq x \leq p\}$$

and let q *be a maximal element of* A.

a) $q \in \Pr E$.

b) $x \in A \Rightarrow xq = qx = x$.

c) q *is the greatest element of* A.

a) We have

$$q \leq q^{\frac{1}{2}} \leq p$$

(Corollary 4.2.7.2), so $q^{\frac{1}{2}} \in A$. Since q is maximal,

$$q = q^{\frac{1}{2}} ,$$

$$q^2 = q ,$$

i.e. $q \in \Pr E$.

b) We have, successively, that

$$0 \leq x^2 \leq p$$

(Corollary 4.2.7.2),

$$0 \leq (p-q)x^2(p-q) \leq (p-q)p(p-q) = p - q$$

(Corollary 4.2.2.3, Corollary 4.2.7.6 a $\Rightarrow$ e & g),

$$q \leq q + (p-q)x^2(p-q) \leq p ,$$

$$(p-q)x^2(p-q) = 0$$

(since $q + (p-q)x^2(p-q) \in A$),

$$\|x(p-q)\|^2 = \|(p-q)x^2(p-q)\| = 0\,,$$

$$x(p-q) = 0\,,$$

$$x = xp = xq$$

(Proposition 4.2.7.1 d $\Rightarrow$ f),

$$x = qx\,.$$

c) follows from a),b), and Proposition 4.2.7.1 b $\Rightarrow$ d. ■

Proposition 4.2.7.9 (0) *Let A be a subset of $E_+^{\#}$, such that both A and $\left\{x^{\frac{1}{2}} \mid x \in A\right\}$ have the same infimum y in E_+. Then $y \in \Pr E$ and*

$$xy = yx = y$$

for every $x \in A$.

Since $y^{\frac{1}{2}}$ is a lower bound of $\left\{x^{\frac{1}{2}} \mid x \in A\right\}$ (Proposition 4.2.4.8),

$$y^{\frac{1}{2}} \leq y\,.$$

Since the reverse inequality also holds (Corollary 4.2.7.2),

$$y^{\frac{1}{2}} = y\,,$$

$$y^2 = y\,,$$

$$y \in \Pr E\,.$$

The last assertion follows from the first one and Corollary 4.2.7.3 a $\Rightarrow$ d. ■

Corollary 4.2.7.10 (0) *If a subset of $\Pr E$ has an infimum in E_+, then this infimum is an orthogonal projection.* ■

Corollary 4.2.7.11 (0) *Take $x \in E_+^{\#}$. If $(x^n)_{n\in\mathbb{N}}$ has an infimum y in E_+, then y is an orthogonal projection and*

$$xy = yx = y\,.$$

$(x^n)_{n\in\mathbb{N}}$ and $(x^{2n})_{n\in\mathbb{N}}$ have the same lower bounds (Corollary 4.2.2.5), so they have the same infimum in E_+ and the assertion now follows from Proposition 4.2.7.9. ■

Proposition 4.2.7.12 (0) *Given $x \in E_+$ and $p, q \in \Pr E$, then following are equivalent:*

a) $x \leq p$, $x \leq q$.

b) $n \in \mathbb{N} \Rightarrow x \leq (pqp)^n$, $x \leq (qpq)^n$.

If $x \in \Pr E$, then these assertions are equivalent to the following ones:

c) $n \in \mathbb{N} \Rightarrow x \leq (pqp)^n$.

d) $n \in \mathbb{N} \Rightarrow x \leq (qpq)^n$.

a $\Rightarrow$ b. We have

$$x = (pqp)^n x (pqp)^n \leq (pqp)^n p (pqp)^n = (pqp)^{2n}$$

for every $n \in \mathbb{N}$ (Proposition 4.2.7.1 a $\Rightarrow$ b, Corollary 4.2.2.3). Since the sequence $((pqp)^n)_{n\in\mathbb{N}}$ is decreasing (Corollary 4.2.2.5),

$$x \leq (pqp)^n$$

for every $n \in \mathbb{N}$. It follows that

$$x \leq (qpq)^n$$

for every $n \in \mathbb{N}$.

b $\Rightarrow$ a. By Corollary 4.2.2.3,

$$x \leq pqp \leq p,$$

$$x \leq qpq \leq q.$$

b $\Rightarrow$ c & d is trivial.

c $\Rightarrow$ a. By Corollary 4.2.2.3,

$$x \leq pqp \leq p$$

so that

$$xp = px = x$$

(Corollary 4.2.7.6 a $\Rightarrow$ b). We deduce

$$xqx = xpqpx = x$$

(Corollary 4.2.7.3 a $\Rightarrow$ e) and

$$x \leq q$$

(Corollary 4.2.7.3 e $\Rightarrow$ a).

d $\Rightarrow$ a follows from c $\Rightarrow$ a. ■

Corollary 4.2.7.13 (0) *Take $p, q \in \Pr E$. If $((pqp)^n)_{n\in\mathbb{N}}$ has an infimum r in E_+, then r is an orthogonal projection and it is the infimum of p and q in E_+.*

By Corollary 4.2.7.11, r is an orthogonal projection. By Proposition 4.2.7.12 c $\Rightarrow$ a, r is a lower bound for p, q.

Let x be a lower bound for p and q in E_+. By Proposition 4.2.7.12 a $\Rightarrow$ b, x is a lower bound for $((pqp)^n)_{n\in\mathbb{N}}$ in E_+. Thus $x \leq r$. Hence r is the infimum of p and q in E_+. ■

Proposition 4.2.7.14 (0) *Suppose E is unital. Take $A \subset E_+^\#$ and put*

$$B := \{1 - x \mid x \in A\}.$$

Then the following are equivalent:

a) *A has a supremum y in $E_+^\#$.*

b) *B has an infimum z in $E_+^\#$.*

If these conditions hold, then

$$y = 1 - z.$$

In particular, for $p, q \in \Pr E$, $p \vee q$ exists iff $(1-p) \wedge (1-q)$ exists and in this case

$$1 - p \vee q = (1-p) \wedge (1-q), \quad p \vee q \leq p + q.$$

First assume a) holds. Take $x \in A$. Then

$$x \leq y \leq 1,$$

$$0 \leq 1 - y \leq 1 - x .$$

Hence $1 - y$ is a lower bound for B in $E_+^{\#}$. Let a be a lower bound for B in $E_+^{\#}$. Take $x \in A$. Then

$$a \leq 1 - x ,$$

$$x \leq 1 - a \in E_+^{\#} .$$

$$y \leq 1 - a ,$$

$$a \leq 1 - y ,$$

$$z = 1 - y .$$

The proof is similar when b) holds. ■

Corollary 4.2.7.15 (0) *Let F be a C^*-subalgebra of E such that for every $x \in F_+^{\#}$, the sequence $(x^n)_{n\in\mathbb{N}}$ has an infimum in E_+ which belongs to F. Take $p, q \in \Pr F$. Then p and q have an infimum in E_+ which belongs to $\Pr F$. If in addition E is unital and F is a unital subalgebra of E, then $\Pr F$ is a lattice and for every $p, q \in \Pr F$ the supremum of p and q in $\Pr F$ is its supremum in $\Pr E$.*

Let r be the infimum of $((pqp)^n)_{n\in\mathbb{N}}$ in E_+. By Corollary 4.2.7.13, r is the infimum of p and q in E_+ and it belongs to $\Pr F$. The last assertion follows from the first one and Proposition 4.2.7.14. ■

Corollary 4.2.7.16 *If E_+ is order complete (order σ-complete), then every subset (countable subset) of $\Pr E$ has an infimum in E_+ which belongs to $\Pr E$. If in addition E is unital, then $\Pr E$ is an order complete (order σ-complete) lattice.*

The assertion follows immediately from Corollaries 4.2.7.15 and 4.2.7.10. ■

Proposition 4.2.7.17 (0) *Let F be a C^*-subalgebra of E with the property that every commutative subset A of $F_+^{\#}$, for which $-A$ is well-ordered, has an infimum in E_+ which belongs to F. Then every nonempty subset of $\Pr F$ has an infimum in E_+ which belongs to $\Pr F$. If in addition F is unital then $\Pr F$ is an order complete lattice and an order faithful set of E.*

Let A be a nonempty subset of $\Pr F$. Let $\mathfrak{B}$ denote the set of subsets B of $\Pr F$ for which $-B$ is well-ordered and every lower bound of A in E_+ is a lower bound of B. Given $B, C \in \mathfrak{B}$, define

$$B \preceq C :\Longleftrightarrow (B \subset C \text{ and } (x \in B, y \in C, x \leq y \Longrightarrow y \in B)).$$

$\preceq$ is an inductive order on $\mathfrak{B}$. $\mathfrak{B}$ therefore has a maximal element B (Zorn's Lemma). By Corollary 4.2.7.6 a $\Rightarrow$ b, B is commutative. Hence, by assumption, B has an infimum p in E_+ which belongs to F. By Corollary 4.2.7.10, $p \in \Pr F$. p is larger than any lower bound for A in E_+.

Take $q \in A$. By Corollary 4.2.7.15, p and q posses an infimum r in E_+ which belongs to $\Pr F$. $B \cup \{r\}$ belongs to $\mathfrak{B}$ and

$$B \preceq B \cup \{r\}.$$

Since B is maximal, we deduce that $r \in B$ and so

$$p \leq r \leq q.$$

Hence p is a lower bound for A. It is thus the infimum of A in E_+.

The last assertion follows immediately from the first one (and Proposition 4.2.7.14). ■

Remark. The lattice $\Pr F$ is not necessarily distributive as shown by the example

$$E := F := \mathcal{L}(\mathbb{K})^2.$$

Proposition 4.2.7.18 *Let F be a maximal Gelfand C^*-subalgebra of E and take $p \in \Pr F$ such that $\{\overset{F}{\widehat{p}} \neq 0\}$ is singleton. Then $pEp \subset F$ if $\mathbb{K} = \mathbb{C}$ and $p(\operatorname{Re} E)p \subset F$ if $\mathbb{K} = \mathbb{R}$.*

Let x' be the unique element of $\{\overset{F}{\widehat{p}} \neq 0\}$.

Take $x \in \operatorname{Re} E$, $y \in F$, and put

$$\alpha := x'(y), \quad z := y - \alpha p.$$

Then

$$x'(z) = x'(y) - \alpha x'(p) = \alpha - \alpha = 0$$

so that

$$zp = pz = 0\,.$$

Thus

$$ypxp = \alpha pxp + zpxp = \alpha pxp = \alpha pxp + pxpz = pxpy\,.$$

Hence $pxp \in F^c$. By Proposition 4.2.2.14 a $\Rightarrow$ b, $pxp \in F$. Hence $p(\operatorname{Re} E)p \subset F$. It follows $pEp \subset F$ if $\mathbb{K} = \mathbb{C}$. ■

Proposition 4.2.7.19 *Let $(p_k)_{k\in\mathbb{N}_n}$ be an increasing family in $E^c \cap \operatorname{Pr} E$ for some $n \in \mathbb{N}$ such that $p_1 = 0$ and $p_n = 1$. Then E is the C^*-direct product of the family*

$$((p_{k+1} - p_k)E(p_{k+1} - p_k))_{k\in\mathbb{N}_{n-1}}\,.$$

Fix $k \in \mathbb{N}_{n-1}$. Then

$$(p_{k+1} - p_k)Ep_k = (p_{k+1} - p_k)p_kE = \{0\}\,,$$

$$p_kE(p_{k+1} - p_k) = Ep_k(pk + 1 - p_k) = \{0\}$$

(Corollary 4.2.7.6 a $\Rightarrow$ c & d). It follows that

$$p_{k+1}Ep_{k+1} = ((p_{k+1} - p_k)E(p_{k+1} - p_k)) \oplus (p_kEp_k)$$

(Corollary 4.2.7.6 a $\Rightarrow$ g). The statement now follows by complete induction. ■

Proposition 4.2.7.20 *Assumme E is the C^*-direct product of the family $(E_\iota)_{\iota\in I}$ of unital simple C^*-algebras. For every $J \subset I$ put*

$$p_J : E \longrightarrow E\,, \quad x \longmapsto e_Jx\,.$$

Then $p_J \in E^c \cap \operatorname{Pr} E$ for every $J \subset I$ and the map

$$\mathfrak{P}(I) \longrightarrow E^c \cap \operatorname{Pr} E\,, \quad J \longmapsto p_J$$

is an isomorphism of ordered sets, where $\mathfrak{P}(I)$ is ordered by inclusion.

It is obvious that $p_J \in E^c \cap \operatorname{Pr} E$ for every $J \subset I$ and that the map

$$\mathfrak{P}(I) \longrightarrow E^c \cap \operatorname{Pr} E, \quad J \longmapsto p_J$$

is injective and increasing (Corollary 4.2.7.6 c $\Rightarrow$ a). So we have only to prove that the above map is surjective.

Take $p \in E^c \cap \operatorname{Pr} E$ and put

$$F := \{x \in E \mid px = x\}, \quad G := \{x \in E \mid px = 0\}.$$

F and G are C^*-subalgebras of E and for every $x \in E$,

$$px \in F, \quad (1-p)x \in G, \quad x = px + (1-p)x.$$

Hence E is the C^*-direct product of F and G. Take $\iota \in I$. Since E_ι is simple, we have $E_\iota \subset F$ or $E_\iota \subset G$, where we identified E_ι canonically with a C^*-subalgebra of E. Put

$$J := \{\iota \in I \mid E_\iota \subset F\}.$$

Then $p = p_J$ and therefore the map

$$\mathfrak{P}(I) \longrightarrow E^c \cap \operatorname{Pr} E, \quad J \longmapsto p_J$$

is surjective. ■

Proposition 4.2.7.21 (0) *Assume E is $\mathbb{C}$-order σ-complete, let $(p_j)_{j\in\mathbb{N}_n}$ be a family in $\operatorname{Pr} E$ with $n \in \mathbb{N}$, and put*

$$x := \frac{1}{n}\sum_{j=1}^{n} p_j.$$

a) $\left(x^{\frac{1}{k}}\right)_{k\in\mathbb{N}}$ *is an increasing sequence in $E_+^\#$. Let p be its supremum in E.*

b) $p \in \operatorname{Pr} E$.

c) p *is the supremum of $(p_j)_{j\in\mathbb{N}_n}$ in $\operatorname{Pr} E$ and in $E_+^\#$.*

Let y be an upper bound of $(p_j)_{j\in\mathbb{N}_n}$ in $E_+^\#$. By Corollary 4.2.7.3 a $\Rightarrow$ b & c,

$$\begin{cases} xy = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} p_j y = \dfrac{1}{n}\sum_{j=1}^{n} p_j = x \\ yx = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} y p_j = \dfrac{1}{n}\sum_{j=1}^{n} p_j = x. \end{cases}$$

Let F be a maximal Gelfand C^*-algebra containing x and y (Corollary 4.1.2.3). Since

$$\|x\| \leq \frac{1}{n} \sum_{j=1}^{n} \|p_j\| \leq 1 ,$$

$x \in F_+^{\#}$. a) follows e.g. via the Gelfand transform of F. By Corollary 4.2.2.20, $p \in F$. Using again the Gelfand transform of F, we deduce $p \in \Pr F \subset \Pr E$ and $p \leq y$.

For $j \in \mathbb{N}_n$,

$$\frac{1}{n} p_j \leq x \leq p ,$$

so that by Corollary 4.2.7.6 h $\Rightarrow$ a,

$$p_j \leq p . \qquad \blacksquare$$

Proposition 4.2.7.22 (0) *Assume E finite-dimensional and let P be the set of minimal elements of* $\Pr E \backslash \{0\}$. *Then for every $p \in \Pr E$ there is a finite subset A of P such that*

$$p = \sum_{q \in A} q ,$$

$$q, r \in A \Longrightarrow qr = \delta_{qr} q .$$

Step 1 $\exists q \in P , q \leq p$

Assume the contrary. By recursion, one may construct a strictly decreasing sequence in $\Pr E \backslash \{0\}$. By Corollary 4.2.7.6, this sequence is linearly independent and this contradicts our assumption that E is finite-dimensional.

Step 2 The assertion

Assume the contrary. By recursion, using Step 1 and Corollary 4.2.7.6 a $\Rightarrow$ g, one may construct a sequence $(q_n)_{n \in \mathbb{N}}$ in P such that

$$q_n \leq p - \sum_{i=1}^{n-1} q_i$$

for every $n \in \mathbb{N}$. Then $\left(p - \sum_{i=1}^{n-1} q_i \right)_{n \in \mathbb{N}}$ is a strictly decreasing sequence in $\Pr E$ (Corollary 4.2.7.6 a $\Rightarrow$ g). By Corollary 4.2.7.6, this sequence is linearly independent and this contradicts our assumption that E is finite-dimensional. $\blacksquare$

4.2.8 Approximate Unit

Proposition 4.2.8.1 (0) *The set*

$$\{x \in E_+ \mid \|x\| < 1\}$$

is upward directed.

Define

$$A := \{x \in E_+ \mid \|x\| < 1\},$$

$$f : [0,1[\longrightarrow \mathbb{K}, \quad \alpha \longmapsto \frac{\alpha}{1-\alpha},$$

$$g : \mathbb{R}_+ \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \frac{\alpha}{1+\alpha}.$$

Then

$$f([0,1[) \subset \mathbb{R}_+, \quad g(\mathbb{R}_+) \subset [0,1[,$$

$$\alpha \in [0,1[\Longrightarrow g(f(\alpha)) = \alpha.$$

Take $x, y \in A$. Then

$$\sigma(x) \subset [0,1[, \quad \sigma(y) \subset [0,1[$$

(Corollary 4.2.1.16 a $\Rightarrow$ c), $f(x), f(y) \in E_+$ (Corollary 4.1.3.5),

$$x = g(f(x)) \leq g(f(x) + f(y)) \in A,$$

$$y = g(f(y)) \leq g(f(x) + f(y)) \in A$$

(Corollaries 4.1.3.11, 4.2.4.7 a), 4.1.3.5, 4.2.1.16 c $\Rightarrow$ a). Hence A is upward directed. ■

Theorem 4.2.8.2 (0) (Segal, 1947) *The upper section filter of*

$$\{x \in E_+ \mid \|x\| < 1\}$$

(Proposition 4.2.8.1), i.e. the filter on E generated by the filter base

$$\{\{x \in E_+ \mid \|x\| < 1, x \geq y\} \mid y \in E_+, \|y\| < 1\}$$

is an approximate unit of E. it is called ***the canonical approximate unit of*** *E.*

Define

$$A := \{x \in E_+ \mid \|x\| < 1\}$$

and let $\mathfrak{F}$ denote the upper section filter of A.

Take $x \in \operatorname{Re} E^{\#}$ and $\varepsilon > 0$. Put $F := E(x)$,

$$K := \{x' \in \sigma(F) \mid |\overset{F}{\widehat{x}}(x')| \geq \varepsilon\}$$

(Corollary 4.1.2.3). Since $\overset{F}{\widehat{x}} \in \mathcal{C}_0(\sigma(F))$ (Theorem 2.4.1.3 b), e)) there is, by Urysohn's Theorem, an $f \in \mathcal{C}_0(\sigma(F))$ such that

$$0 \leq f < 1, \quad f \geq 1 - \varepsilon \text{ on } K.$$

Take $z \in E$ with $\overset{F}{\widehat{z}} = f$ (Corollary 4.1.2.5). Then $z \in A$ and

$$\|x - xz\| = \|\overset{F}{\widehat{x}} - \overset{F}{\widehat{x}}\,\overset{F}{\widehat{z}}\| = \|\overset{F}{\widehat{x}}(1 - f)\| \leq \varepsilon$$

(Corollary 4.1.3.5, Theorem 4.1.3.1 b)).

Take $y \in A$ with $z \leq y$. Then

$$0 \leq 1 - y \leq 1 - z$$

(Corollary 4.2.1.17 a $\Rightarrow$ c),

$$x(1 - y)x \leq x(1 - z)x$$

(Corollary 4.2.2.3), so that

$$\|x(1 - y)x\| \leq \|x(1 - z)x\|$$

(Corollary 4.2.1.18) and

$$\|x - xy\|^2 = \|x(1 - y)\|^2 = \|x(1 - y)^{\frac{1}{2}}(1 - y)^{\frac{1}{2}}\|^2 \leq$$

$$\leq \|x(1 - y)^{\frac{1}{2}}\|^2 = \|x(1 - y)^{\frac{1}{2}}(x(1 - y)^{\frac{1}{2}})^*\| = \|x(1 - y)x\| \leq$$

$$\leq \|x(1 - z)x\| \leq \|x - xz\|\,\|x\| \leq \varepsilon.$$

Hence

$$\lim_{y,\mathfrak{F}} xy = x.$$

It follows that

$$x = \lim_{y,\mathfrak{F}} xy = \lim_{y,\mathfrak{F}} yx .$$

Now let x be arbitrary. We have, for every $y \in \operatorname{Re} E$,

$$\|yx - x\|^2 = \|(yx - x)(yx - x)^*\| = \|(yx - x)(x^*y - x^*)\| =$$

$$= \|yxx^*y - xx^*y - yxx^* + xx^*\| .$$

By the above considerations,

$$0 = \|xx^* - xx^* - xx^* + xx^*\| = \lim_{y,\mathfrak{F}} \|yxx^*y - xx^*y - yxx^* + xx^*\| =$$

$$= \lim_{y,\mathfrak{F}} \|yx - x\|^2 ,$$

so that

$$\lim_{y,\mathfrak{F}} yx = x ,$$

$$x = x^{**} = \lim_{y,\mathfrak{F}} (yx^*)^* = \lim_{y,\mathfrak{F}} xy . \qquad \blacksquare$$

Remark. The above theorem shows that the state space $\tau(E)$ and the pure state space $\tau_0(E)$ are well-defined for each C^*-algebra E.

Corollary 4.2.8.3 (0) *There is a unique order relation on E', which renders E' an ordered Banach space and for which E'_+ is the set of positive linear forms on E. We have*

$$\|x' + y'\| = \|x'\| + \|y'\|$$

for all $x', y' \in E'_+$. Moreover, E' is order complete and every operator defined on E' is order continuous.

By Theorem 4.2.8.2, E is quasiunital. By Corollary 4.2.2.13 and Proposition 2.3.4.13 b), there is a unique order relation on E', with respect to which E' is an ordered Banach space and E'_+ is the set of positive linear forms on E. By Corollary 2.3.4.14,

$$\|x' + y'\| = \|x'\| + \|y'\|$$

for all $x', y' \in E'_+$.

The last assertion follows from the final assertion of Proposition 1.7.2.5. $\blacksquare$

Corollary 4.2.8.4 *If* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$, *then* $E'' = E'^{\pi} = E'^{\sigma}$ $((E'' \subset E'^{\pi} \subset E'^{\sigma}, \operatorname{Re} E'^{\sigma} \subset E'')$. *Moreover, the following are equivalent for every nonempty upward directed set* A' *of* E'_+:

a) A' *is bounded above.*

b) A' *is bounded in norm.*

c) *the upper section filter of* A' *converges.*

d) *the upper section filter of* A' *has a point of adherence.*

e) A' *has a supremum.*

If these conditions are fulfilled, then the upper section filter of A' *converges to the supremum* x' *of* A' *and*

$$x'(x) = \sup_{y' \in A'} y'(x)$$

for every $x \in E_+$. *In particular, every closed set of* E' *is order complete and an order faithful set of* E'.

By Theorem 4.2.8.2, E is quasiunital. Hence, except for $\operatorname{Re} E'^{\sigma} \subset E''$ all the claims follow from Proposition 2.3.4.16. By Corollary 4.1.2.7 d), the absolute convex hull of $E'^{\#}_+$ is a 0–neighbourhood in $\operatorname{Re} E'$, so that

$$\operatorname{Re} E'^{\sigma} \subset E''$$

by Proposition 1.7.2.4 b). ■

Remark. In the real case, we have not $E'^{\sigma} \subset E''$ in general. Take an arbitrary linear form on $\{x' \in E' \mid x'^{*} = -x'\}$ and extend it by 0 on $\operatorname{Re} E'$. This linear form belongs to E'^{σ}, but it need not be continuous if E' is infinite–dimensional.

Corollary 4.2.8.5 (0) *Let* $E \neq \{0\}$ *be a* C^**-algebra and take* $x \in \operatorname{Sn} E$.

a) *The closed convex hull of* $\sigma(x) \cup \{0\}$ *and the closed convex hull of* $\overset{\vee}{x}(\tau(E)) \cup \{0\}$ *coincide.*

b) *The extreme points of* $\sigma(x)$ *are contained in* $\overset{\vee}{x}(\tau_0(E)) \cup \{0\}$.

c) *There is an* $x' \in \tau_0(E)$ *with*

$$|x'(x)| = \|x\|$$

d) $\|x\| = \|\overset{\vee}{x}\| = \|\overset{\vee}{x}|\tau_0(E)\|$.

By Corollary 4.1.2.7 e), $\tau(E) \neq \emptyset$ so $\tau_0(E) \neq \emptyset$ (Theorem of Krein–Milman).

Let F be the unital C^*-algebra associated to E (Theorem 4.1.1.13). Then

$$\sigma_E(x) = \sigma_F(x).$$

a) By Corollary 4.1.2.7 b), the closed convex hull of $\sigma_E(x)$ is $\overset{\vee}{x}(\tau(F))$. The assertion now follows from Proposition 2.3.5.5 a).

b) By Proposition 4.1.2.7 b), the extreme points of $\sigma(x)$ are contained in $\overset{\vee}{x}(\tau_0(F))$. The assertion now follows from Proposition 2.3.5.5 b).

c) The assertion follows from b) (and Theorem 4.1.1.16 b)).

d) follows from c) (and Proposition 2.3.5.8). ■

Corollary 4.2.8.6 *Assume* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$. *Given* $x \in E$ $(x \in \operatorname{Re} E)$, $\sigma(x)$ *is contained in the convex hull of* $\overset{\vee}{x}(\tau(E)) \cup \{0\}$.

Let F be the unital C^*-algebra associated to E (Theorem 4.1.1.13). By Corollary 4.1.2.7 a),b),

$$\sigma_E(x) = \sigma_F(x) \subset \overset{\vee}{x}(\tau(F))$$

and by Proposition 2.3.5.5 a), $\overset{\vee}{x}(\tau(F))$ is the convex hull of $\overset{\vee}{x}(\tau(E)) \cup \{0\}$. ■

Corollary 4.2.8.7 *Assume* $\mathbb{K} = \mathbb{C}$. *Given* $x \in E$, *the following are equivalent:*

a) $x \in \operatorname{Re} E$ *(resp.* $x \in E_+$*)*.

b) $\overset{\vee}{x}(\tau(E)) \subset \mathbb{R}$ *(resp.* $\overset{\vee}{x}(\tau(E)) \subset \mathbb{R}_+$*)*.

c) $\overset{\vee}{x}(\tau_0(E)) \subset \mathbb{R}$ *(resp.* $\overset{\vee}{x}(\tau_0(E)) \subset \mathbb{R}_+$*)*.

a $\Rightarrow$ b follows from Corollary 4.2.8.5 a) (and Corollary 4.1.2.13 a $\Rightarrow$ b).

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. Since

$$\overbrace{\operatorname{im} x}^{\vee}|\tau_0(E) = 0,$$

it follows from Corollary 4.2.8.5 d) that

$$\operatorname{im} x = 0.$$

Hence $x \in \operatorname{Re} E$ and the claim now follows from Corollary 4.2.8.5 b). ■

Corollary 4.2.8.8 (0) *Every finite-dimensional C^*-algebra possesses a unit.*

This follows from Proposition 2.2.7.25 b) and Theorem 4.2.8.2. ■

Corollary 4.2.8.9 *Let F be a C^*-subalgebra of E. Then, given any $y' \in F'_+$, there is an $x' \in E'_+$ with*

$$x'|F = y', \quad \|x'\| = \|y'\|.$$

We may assume E to be unital (Theorem 4.1.1.13). First suppose that F is unital. By Corollary 2.3.2.23, there is an $x' \in \operatorname{Re} E'$ such that

$$x'|F = y', \quad \|x'\| = \|y'\|.$$

By Corollary 4.1.2.8 b), x' is positive.

Now suppose that F is not unital. Define

$$G := \{\alpha 1 + x \mid (\alpha, x) \in \mathbb{K} \times F\}.$$

G is a unital C^*-subalgebra of E. By Corollary 2.3.4.12, there is a $z' \in G'_+$ such that

$$z'|F = y', \quad \|z'\| = \|y'\|.$$

By the above considerations, there is an $x' \in E'_+$ with

$$x'|G = z', \quad \|x'\| = \|z'\|.$$

Then

$$x'|F = z'|F = y', \quad \|x'\| = \|y'\|.$$ ■

Corollary 4.2.8.10 *Let F be a C^*-subalgebra of E. Then for each $y' \in \tau_0(F)$, there is an $x' \in \tau_0(E)$ with $x'|F = y'$.*

We may assume that E is unital (Theorem 4.1.1.13, Proposition 2.3.5.5 b)). Put

$$A' := \{x' \in \tau(E) \mid x'|F = y'\}.$$

By Corollary 4.2.8.9, A' is nonempty. Take $x'_1, x'_2 \in \tau(E)$ and $\alpha, \beta \in]0,1[$ with

$$\alpha + \beta = 1, \quad \alpha x'_1 + \beta x'_2 \in A'.$$

Then

$$1 = \alpha + \beta \geq \alpha \|x_1'|F\| + \beta \|x_2'|F\| \geq \|(\alpha x_1' + \beta x_2')|F)\| = 1$$

(Corollary 4.2.8.3), so that

$$\|x_1'|F\| = \|x_2'|F\| = 1\,,$$

$$x_1'|F = x_2'|F \in \tau(F)\,.$$

Since

$$\alpha(x_1'|F) + \beta(x_2'|F) = y'\,,$$

it follows that

$$x_1'|F = x_2'|F = y'\,,$$

$$x_1'\,,\ x_2' \in A'\,.$$

Hence A' is a face of $\tau(E)$. Since A' is obviously closed in $\tau(E)$ and since $\tau(E)$ is compact (Proposition 2.3.5.9 a)), by the Theorem of Krein–Milman (Theorem 1.3.1.10 b)), it has a pure state. ■

Corollary 4.2.8.11 *If E is a Gelfand C^*-algebra, then $\tau_0(E) = \sigma(E)$.*

E is symmetric (Theorem 4.1.2.1) and quasiunital (Theorem 4.2.8.2), the Gelfand transform of E is bijective (Corollary 4.1.2.5), and $\operatorname{Re} E' = E'_+ - E'_+$ (Corollary 4.1.2.7 d)). Hence the assertion follows from Proposition 2.4.2.11 h $\Rightarrow$ b. ■

Proposition 4.2.8.12 *Assume $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). Take $x \in E$ and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E.*

a) *If $(\langle x_n, x'\rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, x'\rangle$) for every $x' \in \tau_0(E)$, then*

$$(\langle x_n, x'\rangle)_{n\in\mathbb{N}}$$

converges (to $\langle x, x'\rangle$) for every $x' \in E'$ ($x' \in \operatorname{Re} E'$).

b) *If $(\langle x_n, x'\rangle)_{n\in\mathbb{N}}$ converges for every $x' \in \tau_0(E)$ and if F is a predual of E, then there is a $y \in E$ such that*

$$\lim_{n\to\infty} \langle x_n, a\rangle = \langle y, a\rangle$$

for each $a \in F$ ($a \in \operatorname{Re} E$).

a) Case 1 E unital and separable

By Proposition 2.3.5.9, $\tau(E)$ is metrizable and compact. By Proposition 2.3.5.8, $(\overset{\vee}{x_n}|\tau_0(E))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{C}(\tau_0(E))$ which converges pointwise to a function f on $\tau_0(E)$ (to $\overset{\vee}{x}|\tau_0(E)$).

Take $x' \in \tau(E)$. By Choquet's Theorem *(G. Choquet, P.-A. Meyer, Existence et unicité des représentations intégrales dans les convexes compacts quelconques, Ann. Inst. Fourier, 13 (1963), Corolaire 14)* there is a positive Radon measure μ on $\tau_0(E)$ such that

$$\int \overset{\vee}{y} d\mu = \langle y, x' \rangle$$

for every $y \in E$ and

$$\mu(\tau(E) \backslash \tau_0(E)) = 0\,.$$

Then, by Lebesgue's Dominated Convergence Theorem,

$$\int f d\mu = \lim_{n\to\infty} \int \overset{\vee}{x_n} d\mu = \lim_{n\to\infty} \langle x_n, x' \rangle$$

$$\left(\langle x, x' \rangle = \int \overset{\vee}{x} d\mu = \lim_{n\to\infty} \int \overset{\vee}{x_n} d\mu = \lim_{n\to\infty} \langle x_n, x' \rangle \right).$$

By Corollary 4.1.2.7 d), if $x' \in E'$ $(x' \in \operatorname{Re} E')$, then $(\langle x_n, x' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, x' \rangle$).

Case 2 E unital

Let F be the unital C^*–subalgebra of E generated by $\{x\} \cup \{x_n | n \in \mathbb{N}\}$. Then F is separable (Corollary 2.3.2.14 c)). Take $y' \in \tau_0(F)$. By Corollary 4.2.8.10, there is an $x' \in \tau_0(E)$ such that $x'|F = y'$. It follows that $(\langle x_n, y' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, y' \rangle$). Given $y' \in F'$, $(y' \in \operatorname{Re} F')$, $(\langle x_n, y' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, y' \rangle$) by Case 1. In particular, $(\langle x_n, x' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, x' \rangle$) whenever $x' \in E'$, $(x' \in \operatorname{Re} E')$.

Case 3 E arbitrary

Let F be the unital C^*–algebra associated to E (Theorem 4.1.1.13). Then $(\langle x_n, y' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, y' \rangle$) whenever $y' \in \tau_0(F)$ (Proposition 2.3.5.5 b)). By Case 2, $(\langle x_n, y' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, y' \rangle$) whenever $y' \in F'$, $(y' \in \operatorname{Re} F')$. Hence $(\langle x_n, x' \rangle)_{n\in\mathbb{N}}$ converges (to $\langle x, x' \rangle$) whenever $x' \in E'$, $(x' \in \operatorname{Re} E')$ (Hahn–Banach Theorem and Corollary 2.3.2.23).

b) By a), $(\langle x_n, a\rangle)_{n\in\mathbb{N}}$ converges whenever $a \in F$ ($a \in \operatorname{Re} F$, by Proposition 2.3.2.22 e)). By the Alaoglu-Bourbaki Theorem, there is a $y \in E$ such that

$$\lim_{n\to\infty} \langle x_n, a\rangle = \langle y, a\rangle$$

for every $a \in F$ ($a \in \operatorname{Re} F$). ■

Theorem 4.2.8.13 (0) (Grothendieck, 1957) *For every $x' \in \operatorname{Re} E'$, there are uniquely determined $x'^+, x'^- \in E'_+$ (Corollary 4.2.8.3), such that*

$$x' = x'^+ - x'^-, \quad \|x'\| = \|x'^+\| + \|x'^-\|.$$

x'^+ and x'^- are called ***the positive*** *and* ***the negative part*** *of x', respectively.*

If $u : E \to F$ is a surjective involutive algebra homomorphism of E onto a C^*-algebra F then

$$(u'y')^+ = u'y'^+, \quad (u'y')^- = u'y'^-$$

for every $y' \in \operatorname{Re} F'$.

The existence was established in Proposition 4.1.2.7 d). We now prove uniqueness. First assume that E is unital. We may assume that $\|x'\| = 1$. Take $y', z' \in E'_+$ with

$$x' = y' - z', \quad \|y'\| + \|z'\| = \|x'\| = 1.$$

Take $\varepsilon > 0$. By Proposition 2.3.2.22 j), there is an $x \in \operatorname{Re} E^\#$, such that

$$x'(x) > 1 - \frac{\varepsilon^2}{8}.$$

Defining

$$y := \frac{1}{2}(1 - x), \quad z := \frac{1}{2}(1 + x),$$

we have that $y, z \in E^\#$ and it follows from

$$-1 \le x \le 1$$

(Corollary 4.2.1.16 a $\Rightarrow$ b) that

$$0 \le y \le 1, \quad 0 \le z \le 1.$$

We have

$$y'(1) + z'(1) = \|y'\| + \|z'\| = 1 < x'(x) + \frac{\varepsilon^2}{8} = y'(x) - z'(x) + \frac{\varepsilon^2}{8}$$

(Corollary 2.3.4.7),

$$y'(y) + z'(z) = \frac{1}{2}(y'(1) - y'(x) + z'(1) + z'(x)) < \frac{\varepsilon^2}{16},$$

$$y'(y) < \frac{\varepsilon^2}{16}, \quad z'(z) < \frac{\varepsilon^2}{16}.$$

Take $a \in E$. We have

$$|y'(ya)|^2 \leq \|y\| \, \|a\|^2 \|y'\| y'(y) < \frac{\varepsilon^2}{16} \|a\|^2,$$

$$|z'(za)|^2 \leq \|z\| \, \|a\|^2 \|z'\| z'(z) < \frac{\varepsilon^2}{16} \|a\|^2$$

(Corollary 4.2.2.6, 4.2.1.17 b $\Rightarrow$ a), so that

$$|y'(ya)| \leq \frac{\varepsilon}{4}\|a\|, \quad |z'(za)| \leq \frac{\varepsilon}{4}\|a\|,$$

$$|x'^{+}(ya)| \leq \frac{\varepsilon}{4}\|a\|, \quad |x'^{-}(za)| \leq \frac{\varepsilon}{4}\|a\|.$$

We deduce

$$|y'(a) - x'^{+}(a)| = |y'((y+z)a) - x'^{+}((y+z)a)| =$$

$$= |y'(ya) + y'(za) - x'^{+}(ya) - x'^{+}(za)| =$$

$$= |y'(ya) - x'^{+}(ya) + z'(za) - x'^{-}(za)| \leq$$

$$\leq |y'(ya)| + |x'^{+}(ya)| + |z'(za)| + |x'^{-}(za)| \leq \varepsilon\|a\|.$$

Since ε is arbitary, it follows that

$$y'(a) = x'^{+}(a)$$

and thus

$$y' = x'^{+}, \quad z' = x'^{-}.$$

Now suppose that E has no unit and let $\mathfrak{F}$ be an approximate unit of E (Theorem 4.2.8.2). Take $y', z' \in E'_+$ with

$$x' = y' - z', \quad \|x'\| = \|y'\| + \|z'\|.$$

Let F be the unital C^*-algebra associated to E. Take $x'_1 \in \operatorname{Re} F'$, $y'_1, z'_1 \in F'_+$ as defined in Proposition 4.1.2.27 a),e). Then

$$x'_1|E = x', \quad y'_1|E = y', \quad z'_1|E = z',$$

$$\|x'_1\| = \|x'\|, \quad \|y'_1\| = \|y'\|, \quad \|z'_1\| = \|z'\|$$

(Proposition 4.1.2.27 c)). Thus

$$y'_1(1) - z'_1(1) = \lim_{x,\mathfrak{F}} y'(x^*x) - \lim_{x,\mathfrak{F}} z'(x^*x) = \lim_{x,\mathfrak{F}} x'(x^*x) = x'_1(1)$$

(Proposition 4.1.2.27 a)), so that

$$x'_1 = y'_1 - z'_1.$$

Moreover,

$$\|x'_1\| = \|x'\| = \|y'\| + \|z'\| = \|y'_1\| + \|z'_1\|$$

and so

$$y'_1 = x'^+_1, \quad z'_1 = x'^-_1,$$

by the first part of the proof. We get

$$y' = x'^+_1|E, \quad z' = x'^-_1|E$$

which proves the uniqueness in this case.

Now we prove the last assertion. By Corollary 4.2.2.11,

$$u'y'^+, u'y'^- \in E'_+.$$

By Theorem 4.2.6.6 and Proposition 1.3.5.2,

$$\|u'y'\| = \|y'\| = \|y'^+\| + \|y'^-\| = \|u'y'^+\| + \|u'y'^-\|.$$

Hence

$$(u'y')^+ = u'y'^+, \quad (u'y')^- = u'y'^-. \qquad \blacksquare$$

Remark. It is obvious that

$$(-x')^+ = x'^-, \quad (-x')^- = x'^+$$

for every $x' \in \operatorname{Re} E'$. In general, the map

$$\operatorname{Re} E' \longrightarrow E', \quad x' \longmapsto x'^+ \text{ (resp. } x'^- \text{)}$$

is not continuous.

Corollary 4.2.8.14 *Let F be a C^*-subalgebra of E and take $y' \in \operatorname{Re} F'$. Then there is an $x' \in \operatorname{Re} E'$ such that*

$$x'|F = y', \quad \|x'\| = \|y'\|,$$

and for any such x',

$$x'^+|F = y'^+, \quad x'^-|F = y'^-.$$

The existence of x' was proved in Corollary 2.3.2.23. Now take $x' \in \operatorname{Re} E'$ with

$$x'|F = y', \quad \|x'\| = \|y'\|.$$

Then

$$y' = x'|F = x'^+|F - x'^-|F,$$

$$\|y'\| = \|x'^+|F - x'^-|F\| \leq \|x'^+|F\| + \|x'^-|F\| \leq \|x'^+\| + \|x'^-\| = \|x'\| = \|y'\|.$$

It follows that

$$\|y'\| = \|x'^+|F\| + \|x'^-|F\|,$$

so that (by Theorem 4.2.8.13)

$$y'^+ = x'^+|F, \quad y'^- = x'^-|F. \qquad \blacksquare$$

Remark. The above relations

$$(x'|F)^+ = x'^+|F, \quad (x'|F)^- = x'^-|F$$

do not hold in general, as shown by the following example

$$E := C(\{0,1\}),$$

$$x' : E \longrightarrow \mathbb{K}, \quad x \longmapsto x(0) - x(1),$$

$$F := \mathbb{K}1.$$

Corollary 4.2.8.15 *Let E be a C^*-algebra without unit, F its associated unital C^*-algebra, and take $x' \in \operatorname{Re} F'$. Then*

$$x'^+|E = (x'|E)^+, \quad x'^-|E = (x'|E)^-.$$

By Theorem 4.2.8.2, E possesses an approximate unit. We use the notation of Proposition 4.1.2.27 in this proof. We have

$$x'_0 = (x'^+)_0 - (x'^-)_0, \quad x'_{01} = (x'^+)_{01} - (x'^-)_{01}$$

and by Proposition 4.1.2.27 f), $(x'^+)_0, (x'^-)_0 \in E'_+$. By Proposition 4.1.2.27 g), we deduce that

$$\|x'\| = \|x'_0\| + \|x' - x'_{01}\| \leq$$

$$\leq \|(x'^+)_0\| + \|(x'^-)_0\| + \|x'^+ - (x'^+)_{01}\| + \|x'^- - (x'^-)_{01}\| =$$

$$= \|x'^+\| + \|x'^-\| = \|x'\|.$$

Thus

$$\|x'_0\| = \|(x'^+)_0\| + \|(x'^-)_0\|,$$

$$(x'|E)^+ = (x'_0)^+ = (x'^+)_0 = x'^+|E,$$

$$(x'|E)^- = (x'_0)^- = (x'^-)_0 = x'^-|E. \qquad \blacksquare$$

Corollary 4.2.8.16 (0) *Let $(x, x') \in \operatorname{Re} E \times E'_+$. If*

$$xx' = x'x,$$

then

$$x^+x' = x'x^+ = (xx')^+, \quad x^-x' = x'x^- = (xx')^-.$$

We may assume E unital (Theorem 4.1.1.13, Proposition 4.1.2.27 e),j)). By the Theorem of Fuglede–Putnam,

$$(x^+)^{\frac{1}{2}}x' = x'(x^+)^{\frac{1}{2}}\,, \quad (x^-)^{\frac{1}{2}}x' = x'(x^-)^{\frac{1}{2}}\,,$$

so that

$$x^+x' = (x^+)^{\frac{1}{2}}x'(x^+)^{\frac{1}{2}} = x'x^+ \in E'_+\,,$$

$$x^-x' = (x^-)^{\frac{1}{2}}x'(x^-)^{\frac{1}{2}} = x'x^- \in E_+$$

(Proposition 2.3.6.4). Since

$$xx' = (x^+ - x^-)x' = x^+x' - x^-x'$$

and

$$\|xx'\| = \| \, |x|x'\| = \|(x^+ + x^-)x'\| = \|x^+x'\| + \|x^-x'\|$$

(Proposition 4.2.6.17, Corollary 4.2.8.3) we deduce

$$x^+x' = (xx')^+\,, \quad x^-x' = (xx')^-$$

(Theorem 4.2.8.13). ■

Example 4.2.8.17 *Take* $n \in \mathbb{N}$. *For* $a \in \mathcal{L}(\mathbb{K}^n)$ *put*

$$\widetilde{a} : \mathcal{L}(\mathbb{K}^n) \longrightarrow \mathbb{K}\,, \quad [\alpha_{ij}]_{i,j\in\mathbb{N}_n} \longmapsto \sum_{i,j=1}^{n} \alpha_{ij}a_{ji}\,.$$

Then for every $a \in \operatorname{Re}\mathcal{L}(\mathbb{K}^n)$,

$$\widetilde{a^+} = \widetilde{a}^+\,, \quad \widetilde{a^-} = \widetilde{a}^-\,.$$

By Example 4.1.4.11 (with the corresponding notation), there is a finite subset A of $\mathbb{K}^n$ and a real function f on A such that

$$\langle x|y\rangle = \delta_{x,y}$$

for all $x, y \in A$ and

$$a = \sum_{x\in A} f(x)[x|x]\,, \quad \sigma(a) = f(A)\,.$$

By Theorem 4.1.4.6 e) and Theorem 4.2.2.9 b),

$$a^+ = \sum_{x \in A} \sup\{f(x), 0\}[x|x]\,,$$

$$a^- = \sum_{x \in A} \sup\{-f(x), 0\}[x|x]\,,$$

By Example 4.1.4.11,

$$\|\widetilde{a}\| = \sum_{x \in A} |f(x)|\,, \quad \|\widetilde{a^+}\| = \sum_{x \in A} \sup\{f(x), 0\}\,,$$

$$\|\widetilde{a^-}\| = \sum_{x \in A} \sup\{-f(x), 0\}\,,$$

$$\|\widetilde{a}\| = \|\widetilde{a^+}\| + \|\widetilde{a^-}\|\,.$$

Since $\widetilde{a^+}$, $\widetilde{a^-}$ are positive (Example 4.2.3.2 a $\Rightarrow$ b), it follows

$$\widetilde{a}^+ = \widetilde{a^+}\,, \quad \widetilde{a}^- = \widetilde{a^-}\,. \qquad \blacksquare$$

Example 4.2.8.18 *Take* $\varphi, \psi \in \mathbb{R}$,

$$x := \begin{bmatrix} \cos^2\varphi & \cos\varphi \sin\varphi \\ \cos\varphi \sin\varphi & \sin^2\varphi \end{bmatrix}\,,$$

$$y := \begin{bmatrix} \cos^2\psi & \cos\psi \sin\psi \\ \cos\psi \sin\psi & \sin^2\psi \end{bmatrix}\,,$$

$$z := x - y\,, \quad \alpha := \varphi + \psi\,, \quad \beta := \varphi - \psi\,,$$

$$u := \frac{1}{2} \begin{bmatrix} \sin\alpha \sin\beta + |\sin\beta| & \sin\beta\cos\alpha \\ \sin\beta\cos\alpha & \sin\alpha\sin\beta + |\sin\beta| \end{bmatrix}\,,$$

$$v := \frac{1}{2} \begin{bmatrix} \sin\alpha \sin\beta + |\sin\beta & -\sin\beta\cos\alpha \\ -\sin\beta\cos\alpha & \sin\alpha\sin\beta + |\sin\beta| \end{bmatrix}\,.$$

Given $a \in \mathcal{L}(\mathbb{K}^2)$, *define*

$$\widetilde{a} : \mathcal{L}(\mathbb{K}^2) \longrightarrow \mathbb{K}\,, \quad [\alpha_{ij}]_{i,j \in \mathbb{N}_2} \longmapsto \sum_{i,j=1}^{2} \alpha_{ij} a_{ji}\,.$$

Then $\widetilde{x}, \widetilde{y} \in \tau_0(\mathcal{L}(\mathbb{K}^2))$ *and*

$$\widetilde{z}^+ = \widetilde{u}\,, \quad \widetilde{z}^- = \widetilde{v}\,.$$

By Example 2.3.5.3 b $\Rightarrow$ a, $\widetilde{x}, \widetilde{y} \in \tau_0(\mathcal{L}(\mathbb{K}^2))$. Now

$$z = \begin{bmatrix} \cos^2\varphi - \cos^2\psi & \cos\varphi \sin\varphi - \cos\psi \sin\psi \\ \cos\varphi \sin\varphi - \cos\psi \sin\psi & \sin^2\varphi - \sin^2\psi \end{bmatrix},$$

so that

$$z^+ = u, \quad z^- = v$$

by Example 4.2.3.7. The assertion now follows from Example 4.2.8.17. ■

Remark. If we take $\psi := -\varphi$ and set

$$\alpha := \begin{cases} 1 & \text{if } \sin 2\varphi \geq 0 \\ -1 & \text{if } \sin 2\varphi < 0 \end{cases},$$

then

$$u = \cos\varphi \sin\varphi \begin{bmatrix} |\alpha| & 1 \\ 1 & |\alpha| \end{bmatrix}, \quad v = \cos\varphi \sin\varphi \begin{bmatrix} |\alpha| & -1 \\ -1 & |\alpha| \end{bmatrix}.$$

Hence if $\frac{2\varphi}{\pi} \notin \mathbb{Z} \cup \left(\mathbb{Z} + \frac{1}{2}\right)$, then

$$x \neq y, \quad u \neq x, \quad v \neq y.$$

This is a counter-example to the assertion "if E is a C^*-algebra and

$$x', y' \in \tau_0(E)$$

are distinct, then

$$(x' - y')^+ = x', \quad (x' - y')^- = y'".$$

Proposition 4.2.8.19 *Let $\mathbb{K} = \mathbb{C}$ and $n \in \mathbb{N}_3$. Every n-dimensional C^*-algebra is isometric to $\ell^\infty(\mathbb{N}_n)$.*

Let E be an n-dimensional C^*-algebra. By Corollary 4.2.8.8, E possesses a unit. By Corollary 4.1.2.5, it is sufficient to show that E is commutative. This is obvious if $n \in \mathbb{N}_2$, so assume $n = 3$. By Corollary 4.1.4.10, there are $p, q \in \Pr E$ so that $\{1, p, q\}$ is an algebraic basis for E. Take $\alpha, \beta, \gamma \in \mathbb{C}$ with

$$pq = \alpha 1 + \beta p + \gamma q\,.$$

Then

$$\alpha 1 + \beta p + \gamma q = pq = p^2 q = \alpha p + \beta p^2 + \gamma pq = \alpha\gamma 1 + (\alpha + \beta + \beta\gamma)p + \gamma^2 q\,,$$

$$\alpha 1 + \beta p + \gamma q = pq = pq^2 = \alpha q + \beta pq + \gamma q^2 = \alpha\beta 1 + \beta^2 p + (\alpha + \gamma + \beta\gamma)q\,,$$

so that

$$\begin{cases} \alpha = \alpha\gamma = \alpha\beta \\ \beta = \alpha + \beta + \beta\gamma = \beta^2 \\ \gamma = \gamma^2 = \alpha + \gamma + \beta\gamma\,. \end{cases}$$

It follows from this system of equations that $\alpha, \beta, \gamma \in \mathbb{R}$. Thus

$$qp = (pq)^* = \overline{\alpha} 1 + \overline{\beta} p + \overline{\gamma} q = pq\,.$$

Hence E is commutative. ■

Remark. a) The above result also follows from Corollary 6.3.6.5.

b) $\mathbb{C}$ is a 2-dimensional real C^*-algebra, wich is not isomorphic to $\ell^\infty(\mathbb{N}_2)$, so the above proposition does not hold for $\mathbb{K} = \mathbb{R}$.

Proposition 4.2.8.20 *Let E be a finite-dimensional complex C^*-algebra. If E has a commutative C^*-subalgebra F of codimension 1, then E is commutative.*

We may assume that $1 \in F$. Put

$$n := \operatorname{Dim} F\,.$$

Using the Gelfand transform, we find a family $(p_i)_{i \in \mathbb{N}_n}$ in $\operatorname{Pr} F$ which forms an algebraic basis for F such that

$$p_i p_j = 0$$

whenever $i, j \in \mathbb{N}_n$ are distinct and

$$\sum_{i=1}^{n} p_i = 1\,.$$

By Corollary 4.1.4.10, there is a $q \in \Pr E$ such that $\{p_i | i \in \mathbb{N}_n\} \cup \{q\}$ is an algebraic basis for E. Take $i \in \mathbb{N}_n$. There is a family $(\alpha_j)_{j \in \mathbb{N}_n}$ in $\mathbb{C}$ and a $\beta \in \mathbb{C}$ such that

$$p_i q = \sum_{j=1}^{n} \alpha_j p_j + \beta q .$$

We get

$$p_i q = p_i p_i q = \alpha_i p_i + \beta p_i q ,$$

$$p_i q p_i = \alpha_i p_i + \beta q p_i .$$

If $\beta = 1$ then $\alpha_i = 0$ and

$$q p_i = p_i q p_i ,$$

$$p_i q = (q p_i)^* = p_i q p_i = q p_i .$$

If $\beta \neq 1$ then

$$p_i q = \frac{\alpha_i}{1 - \beta} p_i$$

and we deduce from Corollary 4.2.7.6 i $\Rightarrow$ b

$$p_i q = q p_i .$$

Hence E is commutative. ■

Proposition 4.2.8.21 *Let E be a unital complex C^*-algebra and take $p \in \Pr E \backslash \{0, 1\}$. If $E(p, 1)$ is a maximal commutative proper C^*-subalgebra of E, then there is a unital C^*-subalgebra of E containing p which is isometric to $\mathcal{L}(\mathbb{C}^2)$.*

Take $x \in \operatorname{Re} E \backslash E(p, 1)$. Assume that

$$px(p - 1) = 0 .$$

Then

$$px = pxp ,$$

so that

$$xp = (px)^* = pxp = px\,.$$

Since the C^*–subalgebra of E generated by $(1, p, x)$ is commutative (Corollary 2.3.2.14 b), this contradicts our hypothesis. Hence

$$y := px(p-1) \neq 0\,.$$

Take $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with

$$\alpha 1 + \beta p + \gamma y + \delta y^* = 0\,.$$

Then

$$0 = p(\alpha 1 + \beta p + \gamma y + \delta y^*)(1-p) = \gamma y\,,$$

$$0 = (1-p)(\alpha 1 + \beta p + \gamma y + \delta y^*)p = \delta y^*\,.$$

Hence

$$\gamma = \delta = 0\,,$$

$$\alpha = \beta = 0\,.$$

Thus $(1, p, y, y^*)$ is a basis for a vector subspace F of E. Since

$$py^*y = 0 = y^*yp\,,$$

the C^*–subalgebra of E generated by $(1, p, y^*y)$ is commutative (Corollary 2.3.2.14 b)). By the hypothesis on $E(p,1)$, it follows that $yy^* \in E(p,1)$. Similarly $y^*y \in E(p,1)$. Hence F is a unital C^*–subalgebra of E containing p.

Let u be the linear map $E \to \mathcal{L}(\mathbb{C}^2)$ defined by

$$u1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad up := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$uy := \|y\| \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad uy^* := \|y\| \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

where we have identified $\mathbb{C}_{2,2}$ with $\mathcal{L}(\mathbb{C}^2)$. It is easy to check that u is a bijective homomorphism of involutive algebras. By Corollary 4.1.1.21, u is an isometry. ■

Proposition 4.2.8.22 *Every 4-dimensional complex C^*-algebra is isometric either to $\ell^\infty(\mathbb{N}_4)$ or $\mathcal{L}(\mathbb{C}^2)$.*

Let E be a 4-dimensional C^*-algebra. By Corollary 4.2.8.8, E possesses a unit. If E is commutative, then the Gelfand transform shows that E is isometric to $\ell^\infty(\mathbb{N}_4)$. Assume that E is not commutative. Take $p \in E\backslash\{0,1\}$ (Corollary 4.1.4.10). By Proposition 4.2.8.20, $E(p,1)$ is a maximal commutative C^*-subalgebra of E. Hence by Proposition 4.2.8.21, E is isometric to $\mathcal{L}(\mathbb{K}^2)$. ■

Remark. The above result also follows from Wedderburn's Theorem (Corollary 6.3.6.5).

Proposition 4.2.8.23 *Let E be a C^*-algebra without unit and $\mathbb{K} \times E$ its associated unital C^*-algebra. Then the map*

$$\mathbb{K} \times E \longrightarrow \mathbb{K} \times E\,, \quad (\alpha, x) \longmapsto (0, x)$$

is a projection of $\mathbb{K} \times E$ onto $\{0\} \times E$ with norm 2 and any such projection has the norm at least 2.

Let u be a projection of $\mathbb{K} \times E$ onto $\{0\} \times E$. We show that $\|u\| \geq 2$. Since $\{0\} \times E$ is an involutive set of $\mathbb{K} \times E$, $\operatorname{re} u$ is also a projection of $\mathbb{K} \times E$ onto $\{0\} \times E$ (Proposition 2.3.2.22 n)). Since

$$\|\operatorname{re} u\| \leq \|u\|$$

(Proposition 2.3.2.7), we may assume that u is selfadjoint. Then there is an $x \in \operatorname{Re} E$ with

$$u(1,0) = (0,x)\,.$$

By Corollary 4.1.4.14, there is a commutative C^*-subalgebra F of E containing x such that

$$\inf_{x' \in \sigma(F)} |x'(x)| = 0\,.$$

Take $x' \in \sigma(F)$. Using the Gelfand transform on F, we find a $y \in F_+^\#$ with

$$x'(y) = 1\,.$$

Then the unital C^*-subalgebra G of $\mathbb{K} \times E$ generated by y is a Gelfand C^*-algebra. Using the Gelfand transform on G, we see that

$$\|(1,-2y)\| \leq 1\,.$$

It follows that

$$\|u\| \geq \|u(1,-2y)\| = \|(0,x-2y)\| = \|x-2y\| \geq$$

$$\geq |x'(x-2y)| = |x'(x)-2| \geq 2-|x'(x)|\,.$$

Since x' is arbitrary, we deduce that

$$\|u\| \geq 2\,.$$

By Proposition 4.1.2.27 b) (and Theorem 4.2.8.2),

$$\|(0,x)\| = \|x\| \leq 2\|(\alpha,x)\|$$

for every $(\alpha,x) \in \mathbb{K} \times E$. Hence the projection of $\mathbb{K} \times E$ onto $\{0\} \times E$

$$\mathbb{K} \times E \longrightarrow \mathbb{K} \times E\,, \quad (\alpha,x) \longmapsto (0,x)$$

has norm at most 2. By the above considerations its norm is exactly 2. ■

Theorem 4.2.8.24 (0) *A real C^*-algebra E for which* $\operatorname{Re} E$ *is one-dimensional is isomorphic to* $\mathbb{R}$,$\mathbb{C}$, *or* $\mathbb{H}$.

Let $\mathfrak{A}$ be the set of finite subsets A of $E\backslash\{0\}$ such that

$$x \in A \Longrightarrow x + x^* = 0\,,$$

$$x,y \in A\,,\ x \neq y \Longrightarrow xy + yx = 0\,.$$

For every $A \in \mathfrak{A}$, denote by F_A the subalgebra of E generated by $A \cup \operatorname{Re} E$. Then F_A is a finite-dimensional involutive subalgebra of E and therefore a C^*-subalgebra of E. By Corollary 4.2.8.8, F_A is unital. Moreover, for every $x \in F_A\backslash\{0\}$, x^*x is a strictly positive element of F_A. Thus there is an $\alpha > 0$ with

$$x^*x = \alpha 1\,.$$

By Proposition 2.3.1.47 e), F_A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Assume E is not isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Then $F_A \neq E$ for every $A \in \mathfrak{A}$. By Proposition 2.3.1.47 c), we can successively find elements x, y, and z of E such that

$$\{x\}, \{x,y\}, \{x,y,z\} \in \mathfrak{A},$$

$$y \notin F_{\{x\}}, \quad z \notin F_{\{x,y\}}.$$

The relation $z \notin F_{\{x,y\}}$ contradicts Proposition 2.3.1.47 d). Hence E is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. ■

Corollary 4.2.8.25 (0) *Let E be a real C^*-algebra. If the dimension of every Gelfand C^*-subalgebra of E is at most 1, then E is isomorphic to $\{0\}, \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.*

Let $x \in \operatorname{Re} E$, $x \neq 0$. Then there is an $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that

$$x^2 = \alpha x.$$

Put

$$y := \frac{1}{\alpha} x.$$

Then

$$y^2 = \frac{1}{\alpha^2} x^2 = \frac{1}{\alpha} x = y.$$

Let x, y be distinct elements of $\operatorname{Re} E \backslash \{0\}$. We want to show that x and y are linearly dependent. By the above we may assume

$$x^2 = x, \quad y^2 = y.$$

There are $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, such that

$$(x+y)^2 = \alpha(x+y), \qquad (x-y)^2 = \beta(x-y).$$

It follows successively

$$\begin{cases} \alpha x + \alpha y = (x+y)^2 = x^2 + y^2 + (xy + yx) = x + y + (xy + yx) \\ \beta x - \beta y = (x-y)^2 = x^2 + y^2 - (xy + yx) = x + y - (xy + yx), \end{cases}$$

$$(\alpha + \beta - 2)x + (\alpha - \beta - 2)y = 0,$$

x and y are linearly dependent. Thus $\operatorname{Re} E$ is at most one-dimensional and the assertion follows from Theorem 4.2.8.24. ■

Remark. The real Banach space ℓ^1 endowed with the multiplication

$$\ell^1 \times \ell^1 \longrightarrow \ell^1\,, \quad (x,y) \longmapsto 0$$

and with the involution

$$\ell^1 \longrightarrow \ell^1\,, \quad x \longmapsto -x$$

is an infinite-dimensional real involutive Banach algebra such that $\operatorname{Re}\ell^1 = \{0\}$. This example shows that we cannot replace "C^*-algebra" by "involutive Banach algebra" in the above corollary.

Theorem 4.2.8.26 (0) *Let E be a real C^*-algebra such that*

$$xy = 0 \Longrightarrow (x = 0 \text{ or } y = 0)$$

for all positive elements x, y of E. Then E is isomorphic to $\{0\}, \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Let F be a Gelfand C^*-subalgebra of E, $F \neq \{0\}$. Suppose $\sigma(F)$ has two distinct points. There are $f, g \in \mathcal{C}_0(\sigma(E))_+$ such that

$$\|f\| = \|g\| = 1\,, \quad fg = 0\,.$$

Then there are positive elements x, y of F with $\widehat{x} = f$, $\widehat{y} = g$ (Corollary 4.2.1.12). We have

$$\widehat{xy} = fg = 0\,,$$

so that $xy = 0$ and therefore

$$x = 0 \text{ or } y = 0\,.$$

This contradicts the relations

$$\|x\| = \|f\| = 1\,, \quad \|y\| = \|f\| = 1\,.$$

Hence $\sigma(F)$ has exactly one point and F is one-dimensional. By the preceding Corollary, E is isomorphic to $\{0\}, \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. ■

Proposition 4.2.8.27 (0) *Let E be an infinite-dimensional involutive real algebra for which $\operatorname{Re}E$ is finite-dimensional. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that:*

1) $n \in \mathbb{N} \Rightarrow x_n + x_n^* = 0$,

2) $n, p \in \mathbb{N}$, $n \neq p \Rightarrow x_n x_p + x_p x_n = 0$,

3) $n, p, q \in \mathbb{N}$, $n < p < q \Rightarrow x_n x_p x_q = 0$,

4) $n \in \mathbb{N} \Rightarrow \begin{cases} x_n \text{ does not belong to the subalgebra of } E \\ \text{generated by } \{x_m | m \in \mathbb{N}_{n-1}\} \cup \operatorname{Re} E. \end{cases}$

By Proposition 2.3.1.22, $\mathbb{K} = \mathbb{R}$.

We denote for every $A \subset E$ by $E(A)$ the subalgebra of E generated by $A \cup \operatorname{Re} E$. The family $(x_n)_{n \in \mathbb{N}_p}$ in E (with $p \in \mathbb{N} \cup \{0, \infty\}$) is called a Clifford family if for every $n \in \mathbb{N}_p$,

$$x_n + x_n^* = 0,$$

$$x_n \notin E(\{x_m | m \in \mathbb{N}_{n-1}\}).$$

If in addition

$$x_m x_n + x_n x_m = 0$$

for all distinct $m, n \in \mathbb{N}_p$, then $(x_n)_{n \in \mathbb{N}_p}$ is called a strong Clifford family.

Step 1 $\begin{cases} \text{If } (x_n)_{n \in \mathbb{N}_p} \text{ is a Clifford family in } E \text{ with} \\ p \in \mathbb{N} \cup \{0\}, \text{ then there is an } x_{p+1} \in E \text{ such} \\ \text{that } (x_n)_{n \in \mathbb{N}_{p+1}} \text{ is a Clifford family in } E \end{cases}$

By Proposition 2.3.1.49, $E(\{x_n | n \in \mathbb{N}_p\})$ is finite-dimensional. Take

$$x \in E \backslash E(\{x_n | n \in \mathbb{N}_p\}).$$

We set

$$x_{p+1} := x - x^*.$$

Then

$$x_{p+1} + x_{p+1}^* = 0.$$

Since

$$x + x^* \in \operatorname{Re} E \subset E(\{x_n | n \in \mathbb{N}_p\}),$$

it follows

$$x_{p+1} \notin E(\{x_n | n \in \mathbb{N}_p\}) .$$

Hence $(x_n)_{n\in\mathbb{N}_{p+1}}$ is a Clifford family.

Step 2 There is a Clifford sequence in E

The assertion follows from Step 1 by induction.

Step 3
$$\begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \text{ is a Clifford sequence in } E \text{ then} \\ \text{for every } p \in \mathbb{N} \text{ there is a Clifford sequence} \\ (y_n)_{n\in\mathbb{N}} \text{ in } E \text{ such that} \\ n \in \mathbb{N}_p \Rightarrow x_n = y_n , \\ n \in \mathbb{N}\backslash\mathbb{N}_p \Rightarrow x_p y_n + y_n x_p = 0 \end{cases}$$

Since

$$x_p x_n + x_n x_p \in \operatorname{Re} E$$

for every $n \in \mathbb{N}$ and since $\operatorname{Re} E$ is finite-dimensional, there is a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_1 = p$ and such that the family

$$(x_p x_n + x_n x_p)_{n\in\mathbb{N}_{k_{j+1}}\backslash\mathbb{N}_{k_j}}$$

is linearly dependent for every $j \in \mathbb{N}$. Hence for every $j \in \mathbb{N}$, there is a family $(\alpha_{jn})_{n\in\mathbb{N}_{k_{j+1}}\backslash\mathbb{N}_{k_j}}$ in $\mathbb{R}$ such that

$$\sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}(x_p x_n + x_n x_p) = 0 , \quad \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}^2 \neq 0 .$$

We set

$$y_n := x_n$$

for all $n \in \mathbb{N}_p$ and

$$y_{p+j} := \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn} x_n$$

for all $j \in \mathbb{N}$. Then for every $j \in \mathbb{N}$,

$$y_{p+j} + y_{p+j}^* = \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}(x_n + x_n^*) = 0 ,$$

$$y_{p+j} \notin E\left(\{y_n | n \in \mathbb{N}_{p+j-1}\}\right),$$

and

$$x_p y_{p+j} + y_{p+j} x_p = \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}(x_p x_n + x_n x_p) = 0.$$

Step 4 There is a strong Clifford sequence in E

The assertion follows from the Steps 2 and 3 by induction.

Step 5 { If $(x_n)_{n\in\mathbb{N}_3}$ is a strong Clifford family in E, then $x_1x_2x_3 \in \operatorname{Re} E$

This follows from

$$(x_1x_2x_3)^* = x_3^*x_2^*x_1^* = -x_3x_2x_1 = x_1x_2x_3.$$

Step 6 {
If $(x_n)_{n\in\mathbb{N}}$ is a strong Clifford sequence in E, then for every $p, q \in \mathbb{N}$, $p < q$, there is a strong Clifford sequence $(y_n)_{n\in\mathbb{N}}$ in E such that
$n \in \mathbb{N}_q \Longrightarrow x_n = y_n$, $\quad n \in \mathbb{N}\backslash\mathbb{N}_q \Longrightarrow x_p x_q y_n = 0$

By Step 5, $(x_v x_q x_{q+n})_{n\in\mathbb{N}}$ is a sequence in $\operatorname{Re} E$. Since $\operatorname{Re} E$ is finite-dimensional, there is a strictly increasing sequence $(k_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_1 = q$ and such that the family

$$(x_p x_q x_n)_{n\in\mathbb{N}_{k_{j+1}}\backslash\mathbb{N}_{k_j}}$$

is linearly dependent for every $j \in \mathbb{N}$. Hence for each $j \in \mathbb{N}$, there is a family $(\alpha_{jn})_{n\in\mathbb{N}_{k_{j+1}}\backslash\mathbb{N}_{k_j}}$ in $\mathbb{R}$ such that

$$\sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn} x_p x_q x_n = 0, \quad \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}^2 \neq 0.$$

We set

$$y_n := x_n$$

for all $n \in \mathbb{N}_q$ and

$$y_{q+j} := \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn} x_n$$

for all $j \in \mathbb{N}$. Then for every $j \in \mathbb{N}$,

$$y_{q+j} + y_{q+j}^* = \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}(x_n + x_n^*) = 0\,,$$

$$y_{q+j} \notin E\left(\{y_n | n \in \mathbb{N}_{q+j-1}\}\right)\,,$$

and

$$x_p x_q y_{q+j} = \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn} x_p x_q x_n = 0\,,$$

$$x_m y_{q+j} + y_{q+j} x_m = \sum_{n=k_j+1}^{k_{j+1}} \alpha_{jn}(x_m x_n + x_n x_m) = 0$$

for all $m \in \mathbb{N}_{k_j} \cup \left(\mathbb{N} \backslash \mathbb{N}_{k_{j+1}}\right)$, and therefore

$$y_{q+i} y_{q+j} + y_{q+j} y_{q+i} = \sum_{n=k_i+1}^{k_{i+1}} \alpha_{in}(x_n y_{q+j} + y_{q+j} x_n) = 0$$

for all $i \in \mathbb{N}$, $i \neq j$.

Step 7 The assertion of the proposition

Follows from the Steps 4 and 6 by induction. ■

Theorem 4.2.8.28 (0) *Let E be a C^*-algebra. If $\operatorname{Re} E$ is finite-dimensional, then E is also finite-dimensional.*

Assume E is infinite-dimensional. By Proposition 4.2.8.27, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that:

1) $n \in \mathbb{N} \Rightarrow x_n + x_n^* = 0$,

2) $n, p \in \mathbb{N}$, $n \neq p \Rightarrow x_n x_p + x_p x_n = 0$,

3) $n, p, q \in \mathbb{N}$, $n < p < q \Rightarrow x_n x_p x_q = 0$,

4) $n \in \mathbb{N} \Rightarrow$ x_n does not belong to the subalgebra of E generated by $\{x_m | m \in \mathbb{N}_{n-1}\} \cup \operatorname{Re} E$.

By 1),

$$x_n^2 = -x_n^* x_n \in \operatorname{Re} E$$

for every $n \in \mathbb{N}$ and by 2),

$$x_n^2 x_p^2 = x_p^2 x_n^2$$

for all $n, p \in \mathbb{N}$. Hence the C^*-subalgebra F of E generated by $\{x_n^2 | n \in \mathbb{N}\}$ is a Gelfand C^*-algebra (Corollary 4.1.2.3). Since $F \subset \operatorname{Re} E$, F is finite-dimensional. Hence $\sigma(F)$ is finite. In particular, F is unital and there are $p, q \in \mathbb{N}$, $p < q$, such that

$$\sum_{n=1}^{p} \widehat{x_n^2}, \quad \sum_{n=p+1}^{q} \widehat{x_n^2}$$

are strictly negative (Proposition 4.2.1.19). Hence

$$\sum_{n=1}^{p} x_n^2, \quad \sum_{n=p+1}^{q} x_n^2$$

are invertible in F. By 2) and 3),

$$\left(\sum_{m=1}^{p} x_m^2\right)\left(\sum_{n=p+1}^{q} x_n^2\right) x_{q+1}^2 = \sum_{m=1}^{p} \sum_{n=p+1}^{q} x_m^2 x_n^2 x_{q+1}^2 = 0\,,$$

so that by 1),

$$x_{q+1}^* x_{q+1} = -x_{q+1}^2 = 0\,,$$

$$\|x_{q+1}\|^2 = \|x_{q+1}^* x_{q+1}\| = 0\,,$$

$$x_{q+1} = 0\,.$$

This contradicts 4). ■

Remark. a) The real Banach space ℓ^1 endowed with the multiplication

$$\ell^1 \times \ell^1 \longrightarrow \ell^1\,, \quad (x, y) \longmapsto 0$$

and with the involution

$$\ell^1 \longrightarrow \ell^1\,, \quad x \longmapsto -x$$

is an infinite-dimensional real involutive Banach algebra with $\operatorname{Re}\ell^1 = \{0\}$. If E denotes the canonical involutive unital real Banach algebra associated to ℓ^1 (Proposition 2.3.2.8), then E is infinite-dimensional and $\operatorname{Re}E$ is one-dimensional. These examples show that the Theorems 4.2.8.24 and 4.2.8.28 do not hold for arbitrary involutive real Banach algebras.

b) In order that E is finite-dimensional it is sufficient to assume that every Gelfand C^*-subalgebra of E is finite-dimensional (Proposition 6.3.6.17 e $\Rightarrow$ a).

Proposition 4.2.8.29 *Let E be a C^*-algebra and*

$$\mathcal{P} := \{p \in E^c \cap \operatorname{Pr}E | Ep \quad \textit{is one-dimensional}\}\,.$$

a) *For every $p \in \mathcal{P}$ there is a unique $x'_p : E \to \mathbb{K}$ such that*

$$xp = x'_p(x)p$$

for every $x \in E$. x'_p is a non-degenerate positive algebra homomorphism.

b) *If E is the C^*-direct sum (or product) of a family of unital simple C^*-algebras, then for every non-degenerate algebra homomorphism $x' : E \to \mathbb{K}$ there is a $p \in \mathcal{P}$ such that*

$$x' = x'_p\,.$$

In this case, there are exactly $\operatorname{Card}\mathcal{P}$ *factors of the form* $\mathbb{K}$ *in* E.

a) The existence and uniqueness of x'_p is obvious. Take $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. Then

$$x'_p(\alpha x + \beta y)p = (\alpha x + \beta y)p = \alpha(xp) + \beta(yp) =$$

$$= \alpha x'_p(x)p + \beta x'_p(y)p = (\alpha x'_p(x) + \beta x'_p(y))p\,,$$

$$x'_p(xy)p = (xy)p = (xp)(yp) = (x'_p(x)p)(x'_p(y))p = (x'_p(x)x'_p(y))p\,,$$

$$x'_p(x^*)p = x^*p = (xp)^* = (x'_p(x)p)^* = \overline{x'_p(x)}p\,,$$

so that x'_p is an involutive algebra homomorphism. It follows that

$$x'_p(x^*x) = \overline{x'(x)}x'(x) \in \mathbb{R}_+$$

for every $x \in E$, i.e. x'_p is positive. Since

$$x'_p(p)p = pp = p\,,$$

it follows that x'_p is non–degenerate.

b) Let E be the C^*–direct sum (or product) of the family $(E_\iota)_{\iota\in I}$ of unital simple C^*–algebra. For every $\iota \in I$ let $\widetilde{E}_\iota$ be the canonical image of E_ι in E. $\operatorname{Ker} x'$ is a closed ideal of E (Proposition 2.2.4.19) and $E/\operatorname{Ker} x'$ is one–dimensional. For every $\iota \in I$, since E_ι is simple, either $\widetilde{E}_\iota \subset \operatorname{Ker} x'$ or $\widetilde{E}_\iota \cap \operatorname{Ker} x' = \{0\}$. Since $E/\operatorname{Ker} x'$ is isomorphic to the C^*–direct sum (or product) of the family $(\widetilde{E}_\iota/\operatorname{Ker} x')_{\iota\in I}$, it follows that there is a unique $\lambda \in I$ such that $\widetilde{E}_\lambda \cap \operatorname{Ker} x' = \{0\}$. Moreover, E_λ is one–dimensional. Then

$$p := (\delta_{\iota,\lambda})_{\iota\in I} \in \mathcal{P}$$

and

$$x'_p = x'\,.$$

The last assertion follows from Proposition 4.2.7.20. ■

4.3 Supplementary Results on C^*-Algebras

The ordering of a C^*-algebra does not give rise to a lattice, even in the simplest non-commutative case of two-by-two matrices. However the situation improves when the order is complete (as in the case of W^*-algebras) and when attention is restricted to orthogonal projections. Then we have a complete lattice. If, in addition, the C^*-algebra is unital and if p is an orthogonal projection, then $1-p$ is the complementary orthogonal projection – that is to say $p \wedge (1-p) = 0$ and $p \vee (1-p) = 1$. (But observe that the orthogonal projections do not form a Boolean algebra because of the failure of distributivity.) If the order is complete, then the functional calculus can be extended significantly for the self-normal elements, with Borel functions in lieu of continuous functions, and every element of the C^*-algebra has a left support and a right support. Even more is true: every self-normal element has an integral representation (by means of a spectral measure), which degenerates to a sum in the case of compact operators on Hilbert spaces. The entire theory takes on some of the features of integration theory in this case. For example, the above mentioned supports coincide in the case of self-normal operators.

In the final part of this section, we discuss the hereditary C^*-subalgebra and the simple C^*-algebras.

In this section, E denotes a C^*-algebra.

4.3.1 The Exterior Multiplication

Proposition 4.3.1.1

$$(x', p) \in E' \times \Pr E \Longrightarrow \|px'(1-p) + (1-p)x'p\| \leq \|x'\| .$$

Take $x \in E^{\#}$. By Proposition 4.1.2.23,

$$\|px(1-p) + (1-p)xp\| = \sup\{\|px(1-p)\| , \|(1-p)xp\|\} \leq 1 .$$

Hence

$$|\langle x, px'(1-p) + (1-p)x'p\rangle| = |\langle (1-p)xp + px(1-p), x'\rangle| \leq \|x'\| ,$$

$$\|px'(1-p) + (1-p)x'p\| \leq \|x'\| . \qquad \blacksquare$$

Proposition 4.3.1.2 *Let F be an involutive E-module and $(x,a) \in E \times \operatorname{Re} F$. If*

$$xa = ax\,,$$

then

$$|x|a = a|x|\,.$$

By Proposition 4.2.5.3, $|x| \in E(x)$. Thus the assertion follows from Proposition 2.3.6.3. ■

Proposition 4.3.1.3 (0) *Take $(x,x') \in E_+ \times E'_+$. If*

$$xx' = x'x\,,$$

then

$$xx' \in E'_+\,, \quad \|xx'\| = x'(x)\,.$$

By the Fuglede–Putnam Theorem and Proposition 2.3.6.4,

$$xx' = x^{\frac{1}{2}}x^{\frac{1}{2}}x' = x^{\frac{1}{2}}x'x^{\frac{1}{2}} \in E'_+\,,$$

$$\|xx'\| = x'\left(x^{\frac{1}{2}}x^{\frac{1}{2}}\right) = x'(x)\,.$$ ■

Proposition 4.3.1.4 (0)

$$(x,x') \in E_+ \times E'_+ \Longrightarrow \|xx'\|^2 \leq x'(x)\|x'\|\,\|x\|\,.$$

Given $y \in E$,

$$|\langle y, xx'\rangle|^2 = |x'(yx)|^2 = \left|x'\left(yx^{\frac{1}{2}}x^{\frac{1}{2}}\right)\right| \leq$$

$$\leq x'(x)x'(yxy^*) \leq x'(x)\|x'\|\,\|x\|\,\|y\|^2$$

(Proposition 2.3.4.6 c)) and so

$$\|xx'\|^2 \leq x'(x)\|x'\|\,\|x\|\,.$$ ■

Corollary 4.3.1.5 (0) *Given $(x,x') \in E_+ \times E'_+$, the following are equivalent:*

a) $x'(x) = 0$.

b) $xx' = 0$.

c) $x'x = 0$.

d) $xx'x = 0$.

a $\Rightarrow$ b follows from Proposition 4.3.1.4.

b $\Rightarrow$ c. $x'x = (xx')^* = 0$.

c $\Rightarrow$ d is trivial.

d $\Rightarrow$ a. If $\mathfrak{F}$ is an approximate unit of E (Theorem 4.2.8.2), then

$$x'(x^2) = \lim_{y,\mathfrak{F}} x'(xyx) = \lim_{y,\mathfrak{F}} \langle y, xx'x \rangle = 0 .$$

By Proposition 4.2.4.12,

$$x'(x) = 0 .$$ ■

Corollary 4.3.1.6 (0) *Let A be a nonempty downward directed set of E_+ and $\mathfrak{F}$ its lower section filter. Take $x' \in E'_+$. Then the following are equivalent:*

a) $\inf_{x \in A} x'(x) = 0$.

b) $\lim_{x,\mathfrak{F}} xx' = 0$.

c) $\lim_{x,\mathfrak{F}} x'x = 0$.

a $\Rightarrow$ b follows from Proposition 4.3.1.4 (and Corollary 4.2.1.18).

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By Theorem 4.2.8.2, E is quasiunital, so the assertion follows from Proposition 2.2.7.20. ■

Definition 4.3.1.7 (0) *Given an E-module F, $\dot{F}$ denotes the set of $a \in F$ such that the maps*

$$E \longrightarrow F , \quad x \longmapsto xa ,$$

$$E \longrightarrow F , \quad x \longmapsto ax$$

are order continuous.

Proposition 4.3.1.8 $E^\pi \subset \dot{E'}$.

Since $\dot{E'}$ is a vector subspace of E', the assertion follows from Corollary 4.3.1.6 a $\Rightarrow$ b & c. ■

Proposition 4.3.1.9 *Let $\mathfrak{F}$ be an approximate unit of E and F an (involutive) E-module such that*

$$\lim_{x,\mathfrak{F}} \|xa\| = \lim_{x,\mathfrak{F}} \|ax\| = \|a\|$$

for every $a \in F$. Then $\dot{F}$ is an (involutive) E-submodule of F.

Take $a \in \dot{F}$. If $x \in E$, then

$$\|(xa)y\| = \|x(ay)\| \leq \|x\| \, \|ay\| ,$$

$$\|y(ax)\| = \|(ya)x\| \leq \|ya\| \, \|x\|$$

for every $y \in E$. Thus the maps

$$E \longrightarrow F , \quad y \longmapsto (xa)y ,$$

$$E \longrightarrow F , \quad y \longmapsto y(ax)$$

are order continuous.

By Proposition 2.2.7.17, we may assume that E and F are unital. Let x be an invertible element of E, A a downward directed set of E with infimum 0, and $\mathfrak{G}$ the lower section filter of A. Then x^*Ax, xAx^* are downward directed (Corollary 4.2.2.3) and have infimum 0 (Proposition 4.2.2.16). Given $y \in E$,

$$\|y(xa)\| = \|(x^*)^{-1}(x^*yx)a\| \leq \|(x^*)^{-1}\| \, \|(x^*yx)a\| ,$$

$$\|(ax)y\| = \|a(xyx^*)(x^*)^{-1}\| \leq \|a(xyx^*)\| \, \|(x^*)^{-1}\| ,$$

so that

$$\lim_{y,\mathfrak{G}} \|y(xa)\| = \lim_{y,\mathfrak{G}} \|(ax)y\| = 0 .$$

Hence $xa, ax \in \dot{F}$. Since every element of E is the differnece of two invertible elements of E, $\dot{F}$ is an E- submodule of F.

It is easy to see that if F is an involutive E-module, then $\dot{F}$ is an involutive set of F. ■

Proposition 4.3.1.10 *Let E be unital, U the topological group of unitary elements of E (Proposition 2.3.1.8, Theorem 2.2.4.14), and $\mathcal{F}$ the topological group of involutive isometries $E' \to E'$ (Example 2.2.2.4). Given $x \in U$, define*

$$\widetilde{x} : E' \longrightarrow E' , \quad x' \longmapsto xx'x^* .$$

a) *If* $x \in U$ *and*

$$u : E \longrightarrow E\,, \quad y \longmapsto x^*yx\,,$$

then

$$u' = \widetilde{x}\,.$$

b) $\widetilde{x} = \widetilde{x}^* \in \mathcal{F}$ *for every* $x \in U$.

c) *The map*

$$U \longrightarrow \mathcal{F}\,, \quad x \longmapsto \widetilde{x}$$

is a continuous group homomorphism.

d) $\widetilde{x}(\tau_0(E)) = \tau_0(E)$ *for every* $x \in U$.

e) *If* $\mathbb{K} = \mathbb{C}$ *then*

$$\{x \in U \mid \widetilde{x} = \textit{identity map}\} = \{x \in U \mid x' \in E' \Longrightarrow xx' = x'x\} =$$

$$= \{x \in U \mid x' \in \tau_0(E) \Longrightarrow xx' = x'x\}\,.$$

a) Take $(y, x') \in E \times E'$. Then

$$\langle uy, x'\rangle = \langle x^*yx, x'\rangle = \langle y, xx'x^*\rangle = \langle y, \widetilde{x}x'\rangle\,,$$

so that $u' = \widetilde{x}$.

b) follows from a) and Proposition 4.1.1.24 (and Corollary 1.3.4.7, Proposition 2.3.2.22 d)).

c) follows from a) and Proposition 4.1.1.24 (and Corollary 1.3.4.3 and 1.3.4.5).

d) Take $x' \in \tau_0(E)$ and $y' \in E'_+$ with

$$y' \leq \widetilde{x}x'\,.$$

Then

$$x^*y'x \leq x'$$

(Proposition 2.3.6.4). By b) and Proposition 2.3.5.4 a $\Rightarrow$ b,

$$x^*y'x = \|x^*y'x\|x' = \|y'\|x'\,,$$

and so

$$y' = \|y'\| xx'x^* = \|y'\| \widetilde{x}x' .$$

By b) and Proposition 2.3.5.4 b $\Rightarrow$ a, $\widetilde{x}x' \in \tau_0(E)$, so that $\widetilde{x}(\tau_0(E)) \subset \tau_0(E))$. By c), $\widetilde{x}(\tau_0(E)) = \tau_0(E)$.

e) Take $x \in U$ with

$$xx' = x'x$$

for ever $x' \in \tau_0(E)$. Then

$$\widetilde{x}x' = x'$$

for every $x' \in \tau_0(E)$. Since the map

$$E'_E \longrightarrow E'_E , \quad x' \longmapsto \widetilde{x}x'$$

is continuous,

$$\widetilde{x}x' = x'$$

for every $x' \in \tau(E)$ by Proposition 2.3.5.10 b). By Corollary 4.1.2.7 d),

$$\widetilde{x}x' = x'$$

for every $x' \in E'$. Hence

$$\{x \in U \mid x' \in \tau_0(E) \Longrightarrow xx' = x'x\} \subset \{x \in U \mid \widetilde{x} = \text{identity map}\} .$$

The other inclusions are trivial. ■

Proposition 4.3.1.11 *Take* $x', y' \in E'_+$ *and* $p \in \Pr E$. *If*

$$x' \leq py'p ,$$

then

$$x' = px'p .$$

Suppose that E is unital. Then

$$0 \leq (1-p)x'(1-p) \leq (1-p)py'p(1-p) = 0$$

(Proposition 2.3.6.4),

$$(1-p)x'(1-p) = 0\,,$$

$$(1-p)x' = x'(1-p) = 0$$

(Corollary 4.3.1.5 d $\Rightarrow$ b & c),

$$x' = px' = x'p\,,$$

$$x' = px'p\,.$$

If, on the other hand, E has no unit, let $\mathbb{K} \times E$ be the unital C^*-algebra associated to E (Theorem 4.1.1.13). We use the notation from Proposition 4.1.2.27 (Theorem 4.2.8.2). By Proposition 4.1.2.27 e), j),

$$0 \leq x_1' \leq (py'p)_1 = py_1'p\,.$$

By the above considerations and Proposition 4.1.2.27 c),j),

$$x_1' = px_1'p = (px'p)_1\,,$$

$$x' = x_{10}' = (px'p)_{10} = px'p\,. \qquad \blacksquare$$

Example 4.3.1.12 *Take $n \in \mathbb{N}$ and let $\mathcal{L}(\mathbb{K}^n)$ be the C^*-algebra defined in Example 4.1.1.7. Given $a \in \mathcal{L}(\mathbb{K}^n)$, define*

$$\widetilde{a} : \mathcal{L}(\mathbb{K}^n) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{i,j=1}^{n} x_{ij}a_{ji}$$

and identify $\mathcal{L}(\mathbb{K}^n)$ with $\mathcal{L}(\mathbb{K}^n)'$ using the map

$$\mathcal{L}(\mathbb{K}^n) \longrightarrow \mathcal{L}(\mathbb{K}^n)'\,, \quad a \longmapsto \widetilde{a}$$

(Example 4.2.3.2). Then

$$\widetilde{ax} = \widetilde{a}x\,, \quad \widetilde{xa} = x\widetilde{a}$$

for all $a, x \in \mathcal{L}(\mathbb{K}^n)$.

Given $y \in \mathcal{L}(\mathbb{K}^n)$,

$$\langle y, \widetilde{a}x\rangle = \langle xy, \widetilde{a}\rangle = \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} x_{ik}y_{kj}\right)a_{ji} = \sum_{j,k=1}^{n}\left(\sum_{i=1}^{n} a_{ji}x_{ik}\right)y_{kj} = \langle y, \widetilde{ax}\rangle\,,$$

$$\langle y, x\widetilde{a}\rangle = \langle yx, \widetilde{a}\rangle = \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} y_{ik}x_{kj}\right)a_{ji} = \sum_{i,k=1}^{n}\left(\sum_{j=1}^{n} x_{kj}a_{ji}\right)y_{ik} = \langle y, \widetilde{xa}\rangle\,.$$

Thus

$$\widetilde{ax} = \widetilde{a}x\,, \quad \widetilde{xa} = x\widetilde{a}\,. \qquad \blacksquare$$

4.3.2 Order Complete C^*–Algebras

Proposition 4.3.2.1 (0) *Let S, T be locally compact spaces, $\varphi : S \to T$ a continuous map, and x a Borel (Baire) function on T. Then $x \circ \varphi$ is a Borel (Baire) function on S.*

Let U be an exact set of T. Then there is a $y \in \mathcal{C}(T)$ such that

$$U = \{y \neq 0\} .$$

It follows from

$$\overset{-1}{\varphi}(U) = \{y \circ \varphi \neq 0\}$$

that $\overset{-1}{\varphi}(U)$ is exact. We deduce that $\overset{-1}{\varphi}(A)$ is a Borel (Baire) set of S whenever A is a Borel (Baire) set of T. Hence $x \circ \varphi$ is a Borel (Baire) function on S. ■

Theorem 4.3.2.2 (0) *Let E be an order σ–complete (unital) Gelfand C^*–algebra. Take $x \in E$. Let $\mathcal{B}$ be the C^*–algebra of bounded Borel functions f on $\sigma(x)$ such that*

$$\lim_{\alpha \to 0} f(\alpha) = 0$$

if E has no unit.

a) *For each $f \in \mathcal{B}$, there is a unique $\widetilde{f} \in E$ such that*

$$\{x' \in \sigma(E) \mid f \circ \widehat{x}(x') \neq \widehat{\widetilde{f}}(x')\}$$

is meager. Furthermore

$$\sigma(\widetilde{f}) \subset \overline{f(\sigma(x))}, \quad \|\widetilde{f}\| \leq \|f\| .$$

b) *The map*

$$\mathcal{B} \longrightarrow E, \quad f \longmapsto \widetilde{f}$$

is a homomorphism of involutive (unital) algebras such that for any sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{B}_+$,

$$\bigwedge_{n \in \mathbb{N}}^{E} \widetilde{f_n} = \widetilde{\bigwedge_{n \in \mathbb{N}}^{\mathcal{B}} f_n} .$$

c) *Given* $\varepsilon > 0$, *there is a finite family* $((\alpha_\iota, p_\iota))_{\iota \in I}$ *in* $\sigma(x) \times \Pr E$ *such that*

$$\iota, \lambda \in I\,,\ \iota \neq \lambda \Longrightarrow p_\iota p_\lambda = 0$$

$$\|x - \sum_{\iota \in I} \alpha_\iota p_\iota\| < \varepsilon\,.$$

d) *If* f *is a lower semicontinuous real function in* $\mathcal{B}$ *then*

$$f \circ \hat{x} \leq \hat{\tilde{f}}\,.$$

a) Since the Gelfand transform on E is an isometry of C^*-algebras (Corollary 4.1.2.5), $\mathcal{C}_0(\sigma(E))$ is order σ-complete. By Theorem 2.4.1.3 e) and Corollary 2.4.1.7 a), the map

$$\sigma(E) \longrightarrow \sigma(x)\,, \quad x' \longmapsto \hat{x}(x')$$

is continuous and converges to 0 at infinity, if $\sigma(E)$ is not compact. By Proposition 4.3.2.1, $f \circ \hat{x}$ is a Baire function on $\sigma(E)$ converging to 0 at infinity, if $\sigma(E)$ is not compact. By Proposition 1.7.2.13 e $\Rightarrow$ b, there is a unique $g \in \mathcal{C}(\sigma(E))$ such that $\{f \circ \hat{x} \neq g\}$ is meager. Since $\sigma(E)$ is a Baire space, $g \in \mathcal{C}_0(\sigma(E))$. Let $\tilde{f}$ be an element of E with $\hat{\tilde{f}} = g$. Then

$$\|\tilde{f}\| = r(\tilde{f}) \leq \|f\|$$

(Theorem 4.1.1.16 b)) and

$$\sigma(\tilde{f}) \subset \hat{\tilde{f}}(\sigma(E)) \cup \{0\} \subset \overline{f \circ \hat{x}(\sigma(E))} = \overline{f(\sigma(x))}$$

(Corollary 2.4.1.7 a), Proposition 1.7.2.13).

b) It is easy to see that the map

$$\mathcal{B} \longrightarrow \mathcal{C}_0(\sigma(E))\,, \quad f \longmapsto \hat{\tilde{f}}$$

is a homomorphism of involutive (unital) algebras. It follows that the map

$$\mathcal{B} \longrightarrow E\,, \quad f \longmapsto \tilde{f}$$

is a homomorphism of involutive (unital) algebras.

Put

$$f := \bigwedge_{n\in\mathbb{N}}^{\mathcal{B}} f_n\,,$$

$$g : \sigma(E) \longrightarrow \mathbb{K}\,, \quad x' \longmapsto \inf_{n\in\mathbb{N}} \widehat{\widetilde{f}}_n(x')\,.$$

From

$$\left\{\widehat{\widetilde{f}} \neq \bigwedge_{n\in\mathbb{N}} \widetilde{f}_n\right\} \subset \left\{\widehat{\widetilde{f}} \neq f \circ \widehat{x}\right\} \cup \left(\bigcup_{n\in\mathbb{N}} \left\{\widehat{\widetilde{f}}_n \neq f_n \circ \widehat{x}\right\}\right) \cup \left\{\bigwedge_{n\in\mathbb{N}}^{E} \widetilde{f}_n \neq g\right\}$$

it follows that

$$\left\{\widehat{\widetilde{f}} \neq \widehat{\bigwedge_{n\in\mathbb{N}} \widetilde{f}_n}\right\}.$$

is meager (Proposition 1.7.2.13 d $\Rightarrow$ c). Since $\sigma(E)$ is a Baire space,

$$\widehat{\widetilde{f}} = \widehat{\bigwedge_{n\in\mathbb{N}} \widetilde{f}_n}\,.$$

Thus

$$\widetilde{f} = \bigwedge_{n\in\mathbb{N}}^{E} \widetilde{f}_n\,.$$

c) Let $\mathfrak{B}$ be the σ-algebra of all Borel sets B of $\sigma(x)$ such that $0 \notin \overline{B}$ and $j = \sigma(x) \to \mathbb{K}$ the inclusion map. There is a finite family $((\alpha_\iota, A_\iota))_{\iota\in I}$ in $\sigma(x) \times \mathfrak{B}$ with

$$\iota, \lambda \in I\,,\ \iota \neq \lambda \Longrightarrow A_\iota \cap A_\lambda = \emptyset\,,$$

$$\left\| j - \sum_{\iota\in I} \alpha_\iota e_{A_\iota} \right\| < \varepsilon\,.$$

Given $\iota \in I$, put

$$p_\iota := \widetilde{e}_{A_\iota}\,.$$

By b), $p_\iota \in \Pr E$ for every $\iota \in I$. By a) and b)

$$\iota, \lambda \in I\,,\ \iota \neq \lambda \Longrightarrow p_\iota p_\lambda = 0\,,$$

$$\left\| x - \sum_{\iota\in I} \alpha_\iota p_\iota \right\| \leq \left\| j - \sum_{\iota\in I} \alpha_\iota e_{A_\iota} \right\| < \varepsilon\,.$$

d) Take $x' \in \sigma(E)$. Put

$$A := \left\{ f \circ \widehat{x} = \widehat{\widehat{f}} \right\}$$

and let $\mathfrak{F}$ be the trace of the neighbourhood filter of x' on A. Since $f \circ \widehat{x}$ is lower semicontinuous,

$$\widehat{\widehat{f}}(x') = \lim_{y', \mathfrak{F}} \widehat{\widehat{f}}(y') = \lim_{y', \mathfrak{F}} f \circ \widehat{x}(y') \geq f \circ \widehat{x}(x') .$$

Hence

$$f \circ \widehat{x} \leq \widehat{\widehat{f}} .$$ ■

Definition 4.3.2.3 (0) *E is called $\mathbb{C}$-**order complete**, ($\mathbb{C}$-**order σ-complete**) if either*

a) $\mathbb{K} = \mathbb{C}$ *and E is order complete (order σ-complete)*

or

b) $\mathbb{K} = \mathbb{R}$ *and $\overset{\circ}{E}$ is order complete (order σ-complete).*

Corollary 4.3.2.4 (0) *Assume that every $x \in \operatorname{Sn} E$ is contained in an order σ-complete Gelfand C^*-subalgebra of E (this holds whenever E is $\mathbb{C}$-order σ-complete). Then, given $x \in \operatorname{Sn} E$ and $\varepsilon > 0$, there is a finite family $((\alpha_\iota, p_\iota))_{\iota \in I}$ in $\sigma(x) \times \operatorname{Pr} E$ such that*

$$\left\| x - \sum_{\iota \in I} \alpha_\iota p_\iota \right\| < \varepsilon .$$

If $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$), then the vector subspace of E generated by $\operatorname{Pr} E$ is dense in E (in $\operatorname{Re} E$).

Let F be an order σ-complete Gelfand C^*-subalgebra of E containing x. By Theorem 4.3.2.2 c), there is a finite family $((\alpha_\iota, p_\iota))_{\iota \in I}$ in $\sigma(x) \times \operatorname{Pr} F$, such that

$$\left\| x - \sum_{\iota \in I} \alpha_\iota p_\iota \right\| < \varepsilon .$$

Obviously $\operatorname{Pr} F \subset \operatorname{Pr} E$.

The last assertion follows.

Suppose that E is $\mathbb{C}$-order σ-complete. By Corollary 4.1.2.3, every self-normal element of E is contained in a maximal Gelfand C^*-subalgebra of E and by Corollary 4.2.2.20, every maximal Gelfand C^*-subalgebra of E is order σ-complete. ■

Corollary 4.3.2.5 (0) *Let E be $\mathbb{C}$-order σ-complete. Take $x \in \operatorname{Sn} E$. Let $j : \sigma(x) \to \mathbb{K}$ be the inclusion map and $\mathcal{B}$ the C^*-algebra of bounded Borel functions f on $\sigma(x)$ such that*

$$\lim_{\alpha \to 0} f(\alpha) = 0$$

whenever E is not unital.

a) *There is a unique homomorphism of involutive algebras $u : \mathcal{B} \to E$ such that*

$$uj = x$$

and

$$u\left(\bigwedge_{n\in\mathbb{N}}^{\mathcal{B}} f_n\right) = \bigwedge_{n\in\mathbb{N}}^{E} (uf_n)$$

for every decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_+$.

b) *$uf = f(x)$ for every $f \in \mathcal{B} \cap \mathcal{C}(\sigma(x))$. Given $f \in \mathcal{B}$, we define*

$$f(x) := uf\,.$$

c) *If $f \in \mathcal{B}$ and if G is an order σ-faithful C^*-subalgebra of E (e.g. a maximal Gelfand C^*-subalgebra of E) containing x, then*

$$\sigma(f(x)) \subset \overline{f(\sigma(x))}\,, \quad \|f(x)\| \le \|f\|_\infty\,,$$

$$\{x\}^c \subset \{f(x)\}^c\,, \quad f(x) \in G\,.$$

and

$$\left\{ \widehat{\overset{G}{f(x)}} \neq f \circ \overset{G}{\widehat{x}} \right\}$$

is meager whenever G is a Gelfand C^*-algebra.

d) *If there is a $p \in \operatorname{Pr} E$ such that $x \in pEp$, then $f(x) \in pEp$ for every $f \in \mathcal{B}$.*

e) *If $f \in \mathcal{B}$ and g is a bounded Borel function on $\overline{f(\sigma(x))}$ with*

$$\lim_{\alpha \to 0} g(\alpha) = 0$$

whenever E is not unital, then

$$g(f(x)) = g \circ f(x)\,.$$

f) *If* $f \in \mathcal{B}$ *and* $\{f \neq 0\}$ *has nonempty interior with respect to* $\sigma(x)$, *then* $f(x) \neq 0$.

g) *If* 0 *is not an isolated point of* $\sigma(x)$, *then there is a sequence* $(p_n)_{n\in\mathbb{N}}$ *in* $\{x\}^c \cap \Pr E \backslash \{0\}$ *such that*

$$p_m p_n = 0$$

for all distinct $m, n \in \mathbb{N}$ *and*

$$x = \sum_{n\in\mathbb{N}} x p_n .$$

h) *If* $\mathbb{K} = \mathbb{C}$ *and* $n \in \mathbb{N}$, *then:*

h_1) *there is a bounded Borel function* f *on* $\sigma(x)$ *such that*

$$\alpha \in \sigma(x) \Longrightarrow f(\alpha)^n = \alpha .$$

h_2) $f(x)^n = x$.

h_3) x *unitary* $\Rightarrow f(x)$ *unitary.*

i) *If* $x \in \operatorname{Re} E$, F *is an* E*-module, and* $a \in F$ *such that* $ax = 0$ *(resp.* $xa = 0$*) and such that the map*

$$E \longrightarrow F, \quad y \longmapsto ay \ (\textit{resp. } ya)$$

is order σ*-continuous then* $af(x) = 0$ *(resp.* $f(x)a = 0$*) for every* $f \in \mathcal{B}$ *such that*

$$0 \in \sigma(x) \Longrightarrow \lim_{\substack{t\to 0 \\ t\in\sigma(x)}} f(t) = 0 .$$

a & b & c & d. Let F be a maximal Gelfand C^*-subalgebra of E containing x (and p in Case d)) (Corollary 4.1.2.3). By Corollary 4.2.2.20, F is an order σ-faithful set of E and it is $\mathbb{C}$-order σ-complete. By Theorem 4.3.2.2 a),b), there is a homomorphism of involutive algebras $u : \mathcal{B} \to F$, such that

$$uf = f(x)$$

for every $f \in C(\sigma(x)) \cap \mathcal{B}$,

$$u\left(\bigwedge_{n\in\mathbb{N}}^{\mathcal{B}} f_n\right) = \bigwedge_{n\in\mathbb{N}}^{E} (uf_n)$$

for every decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_+$, and

$$\sigma(uf) \subset \overline{f(\sigma(x))}, \quad \|uf\| \leq \|f\|$$

for every $f \in \mathcal{B}$.

Let $v : \mathcal{B} \to E$ be a homomorphism of involutive algebras with the property described in a), and

$$\mathcal{F} := \left\{ f \in \mathcal{B} \;\middle|\; \begin{array}{l} uf = vf \in G, \quad \{x\}^c \subset \{uf\}^c, \\ \\ G \text{ Gelfand} \Longrightarrow \left\{ \overset{G}{\widehat{f(x)}} \neq f \circ \overset{G}{\widehat{x}} \right\} \text{ is meager} \end{array} \right\}$$

Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing (increasing) sequence in $\mathcal{F}$ with infimum (supremum) f in $\mathcal{B}$. Then

$$uf = vf \in G$$

and

$$\{x\}^c \subset \bigcap_{n\in\mathbb{N}} \{uf_n\}^c = \{uf_n \mid n \in \mathbb{N}\}^c \subset \{uf\}^c$$

by Proposition 4.2.2.17. Hence, by Proposition 1.7.2.13 e $\Rightarrow$ c, $f \in \mathcal{F}$. Since

$$\mathcal{C}(\sigma(x)) \cap \mathcal{B} \subset \mathcal{F}$$

(Theorem 4.1.3.1, Corollary 4.1.3.5), we get $\mathcal{F} = \mathcal{B}$.

In order to prove $f(x) \in pEp$ in Case d), we may assume f real and $|f| \leq 1$. Then

$$|f(x)| \leq p$$

and we get

$$f(x) = pf(x)p \in pEp$$

by Proposition 4.2.7.1 a $\Rightarrow$ c.

e) We first remark that, by b) and c), $g(f(x))$ and $g \circ f(x)$ are well–defined. Let $\mathcal{B}_0$ be the C^*–algebra of bounded Borel functions g on $\overline{f(\sigma(x))}$ such that

$$\lim_{\alpha\to 0} g(\alpha) = 0$$

whenever E is not unital and let

$$\mathcal{G} := \{g \in \mathcal{B}_0 \mid g(f(x)) = g \circ f(x)\}\,.$$

Since the inclusion map

$$\overline{f(\sigma(x))} \longrightarrow \mathbb{K}$$

belongs to $\mathcal{G}$ and since the maps

$$\mathcal{B}_0 \longrightarrow E\,, \quad g \longmapsto g(f(x))\,,$$

$$\mathcal{B}_0 \longrightarrow E\,, \quad g \longmapsto g \circ f(x)$$

are homomorphisms of involutive algebras,

$$\mathcal{B}_0 \cap \mathcal{C}(\sigma(f(x))) \subset \mathcal{G}\,.$$

Let $(g_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{G}$ and let g be its infimum in $\mathcal{B}_0$. Then

$$g(f(x)) = \bigwedge_{n\in\mathbb{N}}^{E} g_n(f(x)) = \bigwedge_{n\in\mathbb{N}}^{E} g_n \circ f(x) = g \circ f(x)$$

by a), so that $g \in \mathcal{G}$. It follows that

$$\mathcal{B}_0 = \mathcal{G}\,.$$

f) First assume that f is positive. There is a $g \in \mathcal{C}(\sigma(x))\backslash\{0\}$ such that $0 \leq g \leq f$. By a),

$$g(x) \leq f(x)$$

and by Theorem 4.1.3.1 b),

$$\|g(x)\| = \|g\| \neq 0\,.$$

Hence

$$g(x) \neq 0\,,$$

so that

$$f(x) \neq 0\,.$$

Now let f be arbitrary. Put

$$g : \overline{f(\sigma(x))} \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto |\alpha|\,.$$

By e) and the above considerations,

$$g(f(x)) = g \circ f(x) \neq 0\,.$$

Hence

$$f(x) \neq 0\,.$$

g) We put

$$r_0 := \|x\|\,.$$

Since 0 is not an isolated point of $\sigma(x)$, there is a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of real numbers with infimum 0 such that for every $n \in \mathbb{N}$,

$$\sigma(x) \cap A_n \neq \emptyset\,,$$

where

$$A_n := \{\alpha \in \mathbb{K} \mid r_n < |\alpha| \leq r_{n-1}\}\,.$$

Given $n \in \mathbb{N}$, put

$$p_n := e_{A_n}^{\sigma(x)}(x)\,.$$

By a) and c), $(p_n)_{n\in\mathbb{N}}$ is a sequence in $\{x\}^c \cap \Pr E$ such that

$$p_m p_n = 0$$

for all distinct $m, n \in \mathbb{N}$. By c),

$$\left\| x - \sum_{k=1}^{n} x p_k \right\| \leq r_n$$

for every $n \in \mathbb{N}$, so

$$x = \sum_{n\in\mathbb{N}} x p_n\,.$$

By f),

$$p_n \neq 0$$

for every $n \in \mathbb{N}$.

h) We set

$$f : \sigma(x) \longrightarrow \mathbb{C}\,, \quad re^{i\theta} \longmapsto r^{\frac{1}{n}} e^{i\frac{\theta}{n}}$$

where $\theta \in]-\pi, \pi]$. By e), $f(x)^n = x$. If x is unitary, then by c) and Corollary 4.1.2.13, we get successively

$$\sigma(x) \subset \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}\,,$$
$$\sigma(f(x)) \subset \overline{f(\sigma(x))} \subset \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}\,,$$
$$f(x) \in \operatorname{Un} E.$$

i) Put

$$\mathcal{B}_0 := \{g \in \mathcal{B} \mid 0 \in \sigma(x) \Longrightarrow \lim_{\substack{t \to 0 \\ t \in \sigma(x)}} g(t) = 0\}\,,$$

$$\mathcal{G} := \{g \in \mathcal{B}_0 \mid ag(x) = 0 \quad (\text{resp. } g(x)a = 0)\}\,.$$

Let $(g_n)_{n \in \mathbb{N}}$ be a decreasing (increasing) sequence in $\mathcal{G}$ with infimum (supremum) g in $\mathcal{B}_0$. By a),

$$ag(x) = \lim_{n \to \infty} ag_n(x) = 0$$

$$(\text{resp. } g(x)a = \lim_{n \to \infty} g_n(x)a = 0)$$

so that $g \in \mathcal{G}$. Since $\mathcal{B}_0 \cap \mathcal{C}(\sigma(x)) \subset \mathcal{G}$ (Theorem 4.1.3.1 f)), it follows $\mathcal{B}_0 \subset \mathcal{G}$, i.e. $f \in \mathcal{G}$. ■

Corollary 4.3.2.6 (0) *If E is $\mathbb{C}$-order σ-complete then $\Pr E$ is upward directed and its upper section filter is an approximate unit of E.*

Take $p, q \in \Pr E$. By Proposition 4.2.8.1, $\frac{1}{2}p$ and $\frac{1}{2}q$ haven an upper bound x in $E_+^{\#}$. Let r be the supremum of $\left(x^{\frac{1}{n}}\right)_{n \in \mathbb{N}}$. By Corollary 4.3.2.5 a),b),c), $r \in \Pr E$. From

$$\frac{1}{2}p \leq r\,, \quad \frac{1}{2}q \leq r$$

(Corollary 4.2.7.2), we deduce

$$p \leq r\,, \quad q \leq r$$

(Corollary 4.2.7.6 h $\Rightarrow$ a). Hence $\Pr E$ is upwards directed.

Take $a \in E$ and $\varepsilon > 0$. By Theorem 4.2.8.2, there is an $x \in E_+$ such that $\|x\| < 1$ and such that

$$y \in E_+ , \ \|y\| < 1 , \ x \leq y \Longrightarrow \|ay - a\| < \frac{\varepsilon}{3} , \ \|ya - a\| < \frac{\varepsilon}{3} .$$

We may assume that

$$(1 - \|x\|)\|a\| < \frac{\varepsilon}{3} .$$

Let F be a maximal Gelfand C^*-subalgebra of E containing x (Corollary 4.1.2.3). Define

$$\delta := \inf \left\{ \frac{1 - \|x\|}{2} , \frac{\varepsilon}{3(1 + \|a\|)} \right\} ,$$

$$f : \sigma(x) \longrightarrow \mathbb{K} , \quad \alpha \longmapsto \begin{cases} 1 & \text{if } \alpha > \delta \\ 0 & \text{if } \alpha \leq \delta , \end{cases}$$

$$g : \sigma(x) \longrightarrow \mathbb{K} , \quad \alpha \longmapsto \begin{cases} 0 & \text{if } \alpha > \delta \\ \alpha & \text{if } \alpha \leq \delta , \end{cases}$$

and (Corollary 4.3.2.5 b))

$$p = f(x) , \quad y := g(x) .$$

Then $p \in \Pr F$, $\|y\| \leq \delta$, and

$$x \leq \|x\| p + y$$

(Corollary 4.3.2.5 a),c)). Take $q \in \Pr E$ with $q \geq p$. Then

$$x \leq \|x\| q + y , \quad \| \|x\| q + y\| \leq \|x\| + \|y\| < 1 ,$$

so that

$$\|a(\|x\| q + y) - a\| < \frac{\varepsilon}{3} , \quad \|(\|x\| q + y)a - a\| < \frac{\varepsilon}{3} .$$

We get

$$\|aq - a\| \leq \|a(\|x\| q + y) - a\| + \|(1 - \|x\|)aq\| + \|ay\| \leq$$

$$\leq \frac{\varepsilon}{3} + (1 - \|x\|)\|a\| + \|a\|\delta < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon ,$$

$$\|qa - a\| \leq \|(\|x\|(q + y)a - a\| + \|(1 - \|x\|)qa\| + \|ya\| \leq$$

$$\leq \frac{\varepsilon}{3} + (1 - \|x\|)\|a\| + \delta\|a\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Hence the upper section filter of $\Pr E$ is an approximate unit of E. ■

Corollary 4.3.2.7 *Let $\mathbb{K} = \mathbb{C}$. Let E be unital and order σ-complete, x a unitary element of E, and $\alpha \in \mathbb{R}$. Then there is a unique element $y \in \operatorname{Re} E$ such that*

$$\sigma(y) \subset [\alpha, \alpha + 2\pi], \quad e^{iy} = x, \quad e_\alpha^{\sigma(y)}(y) = 0.$$

We have

$$y = f(x),$$

where

$$f : \sigma(x) \longrightarrow \mathbb{C}, \quad e^{i\theta} \longmapsto \theta \in]\alpha, \alpha + 2\pi].$$

In particular, the set of unitary elements is path-connected.

Now

$$\sigma(x) \subset \{\beta \in \mathbb{C} \mid |\beta| = 1\}$$

(Corollary 4.1.2.13 a $\Rightarrow$ b). Hence f is well-defined. Put

$$y := f(x),$$

$$g : [\alpha, \alpha + 2\pi] \longrightarrow \mathbb{C}, \quad \beta \longmapsto e^{i\beta}.$$

By Corollary 4.3.2.5 c),

$$\sigma(y) = \sigma(f(x)) \subset \overline{f(\sigma(x))} \subset [\alpha, \alpha + 2\pi].$$

Hence $y \in \operatorname{Re} E$ (Corollary 4.1.2.13 b $\Rightarrow$ a). Since

$$\beta \in \sigma(x) \Longrightarrow g \circ f(\beta) = \beta,$$

and

$$e_\alpha^{\sigma(y)} \circ f = 0,$$

we get, by Collary 4.3.2.5 e),

$$e^{iy} = g(y) = g(f(x)) = g \circ f(x) = x,$$

$$e_\alpha^{\sigma(y)}(y) = e_\alpha^{\sigma(y)}(f(x)) = (e_\alpha^{\sigma(y)} \circ f)(x) = 0.$$

We now prove the uniqueness. Take $y \in \operatorname{Re} E$ with

$$\sigma(y) \subset [\alpha, \alpha + 2\pi], \quad e^{iy} = x, \quad e_\alpha^{\sigma(y)}(y) = 0.$$

Given $\beta \in [\alpha, \alpha + 2\pi]$,

$$(f \circ g - 2\pi e_\alpha^{\sigma(y)})(\beta) = \beta,$$

where g is the function defined above. By Corollary 4.3.2.5 e),

$$y = (f \circ g - 2\pi e_\alpha^{\sigma(y)})(y) = f(g(y)) - 2\pi e_\alpha^{\sigma(y)}(y) = f(x).$$

The last assertion follows from the fact that

$$[0, 1] \longrightarrow E, \quad \alpha \longmapsto e^{\alpha i y}$$

is a continuous path in the set of unitary elements of E connecting 1 to x. ■

Corollary 4.3.2.8 *Let $\mathbb{K} = \mathbb{C}$. Let E be unital and order σ-complete, x an invertible element of E, and $\alpha \in \mathbb{R}$. Then there are uniquely determined $y, z \in \operatorname{Re} E$ such that*

$$\sigma(y) \subset [\alpha, \alpha + 2\pi], \quad x = e^{iy}e^z, \quad e_\alpha^{\sigma(y)}(y) = 0.$$

If, in addition, x is normal, then

$$yz = zy, \quad x = e^{z+iy}, \quad z + iy = f(x),$$

where

$$f : \sigma(x) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \log \alpha \in \mathbb{R} \times]\alpha, \alpha + 2\pi].$$

By Theorem 4.2.6.9, $|x|$ is invertible and there is a unique $y_0 \in \operatorname{Un} E$ such that

$$x = y_0|x|.$$

By Corollary 4.3.2.7, there is a unique element $y \in \operatorname{Re} E$ such that

$$\sigma(y) \subset [\alpha, \alpha + 2\pi], \quad e^{iy} = y_0, \quad e_\alpha^{\sigma(y)}(y) = 0,$$

and by Proposition 4.2.4.5 a),b), there is a unique element $z \in \operatorname{Re} E$ such that

$$|x| = e^z.$$

If x is normal, then

$$y_0|x| = |x|y_0$$

(Proposition 4.2.6.12). Thus

$$yz = zy\,, \quad x = e^{z+iy}$$

(Corollary 4.3.2.5 c), Proposition 2.2.3.7 a)). By Corollary 4.3.2.5 e),

$$x = e^{f(x)}$$

and so

$$z + iy = f(x)\,. \qquad \blacksquare$$

Corollary 4.3.2.9 *Let $\mathbb{K} = \mathbb{C}$. If E is unital and order σ-complete, then the set of invertible elementes of E is path-connected.*

Let x be an invertible element of E. By Corollary 4.3.2.8, there are $y, z \in \operatorname{Re} E$ such that

$$x = e^{iy}e^{z}\,.$$

The map

$$[0,1] \longrightarrow E\,, \quad \alpha \longmapsto e^{i\alpha y}e^{\alpha z}$$

is a continuous path in the set of invertible elements of E connecting 1 with x. $\blacksquare$

Lemma 4.3.2.10 (0) *Given $n \in \mathbb{N}$, define*

$$A_n := \bigcup_{k=1}^{2^{n-1}} \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right[\,,$$

$$f_n := \sum_{j=1}^{n} \frac{1}{2^j} e_{A_j}^{[0,1]}\,.$$

a) *Given $n, k \in \mathbb{N}$ and $\alpha \in [0,1]$ such that*

$$\frac{k-1}{2^n} \leq \alpha < \frac{k}{2^n}\,,$$

we have that

$$f_n(\alpha) = \frac{k-1}{2^n}\,.$$

b) *For every* $n \in \mathbb{N}$ *and* $\alpha \in [0,1]$,

$$0 \leq \alpha - f_n(\alpha) < \frac{1}{2^n}.$$

c) $(f_n)_{n\in\mathbb{N}}$ *is an increasing sequence of positive real functions on* $[0,1]$ *converging uniformly to the identity map of* $[0,1]$.

a) We prove the assertion by induction on n. The assertion holds for $n = 1$. Assume it holds up to $n-1$ and take $j \in \mathbb{N}$ with

$$\frac{2j-2}{2^n} = \frac{j-1}{2^{n-1}} \leq \alpha < \frac{j}{2^{n-1}} = \frac{2j}{2^n}.$$

Then $k \in \{2j-1, 2j\}$ and

$$f_{n-1}(\alpha) = \frac{2j-2}{2^n}.$$

If $k = 2j-1$, then $\alpha \notin A_n$ and therefore

$$f_n(\alpha) = f_{n-1}(\alpha) = \frac{2j-2}{2^n} = \frac{k-1}{2^n}.$$

If $k = 2j$, then $\alpha \in A_n$ and therefore

$$f_n(\alpha) = f_{n-1}(\alpha) + \frac{1}{2^n} = \frac{2j-2}{2^n} + \frac{1}{2^n} = \frac{2j-1}{2^n} = \frac{k-1}{2^n}.$$

b) and c) follw from a). ■

Corollary 4.3.2.11 (0) *Let* E *be* $\mathbb{C}$*-order* σ*-complete and take* $x \in E_+^{\#}$. *Given* $n \in \mathbb{N}$, *put*

$$A_n := \sigma(x) \cap \bigcup_{k\in\mathbb{N}} \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right[,$$

$$p_n := e_{A_n}^{\sigma(x)}(x).$$

(Corollary 4.3.2.5 b)). Then $(p_n)_{n\in\mathbb{N}}$ *is a sequence in* $\Pr E$ *and*

$$\left(\sum_{k=1}^{n} \frac{1}{2^k} p_k\right)_{n\in\mathbb{N}}$$

is an increasing sequence in E_+ *converging to its supremum* x. *In particular, every element of* $\operatorname{Re} E'$ *is positive whenever it is positive on* $\Pr E$.

The first assertion follows immediately from Lemma 4.3.2.10 and Corollary 4.3.2.5 a),b). The final assertion follows from Corollary 4.2.2.10. ■

Corollary 4.3.2.12 *If E is $\mathbb{C}$-order σ-complete and $p \in \Pr E\backslash\{0\}$, then the following are equivalent:*

a) *p is a minimal element of $\Pr E\backslash\{0\}$.*

b) *If F is an order σ-complete Gelfand C^*-subalgebra of E containing p, then p is a minimal element of $\Pr F\backslash\{0\}$.*

c) *If F is a maximal Gelfand C^*-subalgebra of E containing p, then p is a minimal element of $\Pr F\backslash\{0\}$.*

d) *If $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$ then pEp $(p(\operatorname{Re} E)p)$ is one-dimensional.*

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c follows from Corollary 4.2.2.20.

c $\Rightarrow$ d. Take $x \in p(\operatorname{Re} E)p$. Assume that $\sigma(x)$ contains at least two points. Let α be a point of $\mathbb{R}$ lying between these two points. We define

$$A := \sigma(x) \cap]-\infty, \alpha[\,, \quad B := \sigma(x) \cap [\alpha, \infty[\,,$$

$$x_A := e_A^{\sigma(x)}(x)\,, \quad x_B := e_B^{\sigma(x)}(x)\,.$$

By Corollary 4.3.2.5 a),b),c),d) (and Corollary 4.2.2.20),

$$x_A, x_B \in (\Pr F\backslash\{0\}) \cap (pEp)\,,$$

$$x_A + x_B = p\,,$$

which contradicts the minimality of p. Hence $\sigma(x)$ contains exactly one point α. We may assume that $\alpha \neq 0$. By Corollary 4.3.2.5 a),b),c),d),

$$p = e_{\sigma(x)}^{\sigma(x)}(x) = \frac{1}{\alpha}x\,,$$

so that

$$x = \alpha p$$

and pEp $(p(\operatorname{Re} E)p)$ is thus one-dimensional.

d $\Rightarrow$ a. Take $q \in \Pr E\backslash\{0\}$ with $q \leq p$. By Corollary 4.2.7.6 a $\Rightarrow$ f,

$$q = pqp \in p(\operatorname{Re} E)p\,.$$

By d),

$$q = p\,,$$

hence p is a minimal element of $\Pr E\backslash\{0\}$. ∎

Lemma 4.3.2.13 (Drewnowski, 1972) *Let $\mathcal{M}$ be a countable set of bounded complex valued functions on $\mathfrak{P}(\mathbb{N})$ such that for each $\mu \in \mathcal{M}$,*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever A and B are disjoint subsets of $\mathbb{N}$. Then there is an infinite subset M of $\mathbb{N}$ such that $\mu|\mathfrak{P}(M)$ is a measure whenever $\mu \in \mathcal{M}$.

We first remark that

$$\sum_{n\in\mathbb{N}} |\mu(\{n\})| < \infty$$

for every $\mu \in \mathcal{M}$ (Proposition 1.1.6.14 d $\Rightarrow$ b). Given $\mu \in \mathcal{M}$, define

$$\widetilde{\mu} : \mathfrak{P}(\mathbb{N}) \longrightarrow \mathbb{C}, \quad A \longmapsto \mu(A) - \sum_{n\in A} \mu(\{n\}).$$

Let $\mathfrak{A}$ be an uncountable set of infinite subsets of $\mathbb{N}$ such that $A \cap B$ is finite for distinct $A, B \in \mathfrak{A}$ (Lemma 1.1.2.17). Let $\mathcal{N}$ be a finite subset of $\mathcal{M}$ and take $\varepsilon > 0$. Let $\mathfrak{B}$ be the set of $A \in \mathfrak{A}$ such that there are $\mu \in \mathcal{N}$ and $B \subset A$ with

$$|\widetilde{\mu}(B)| > \varepsilon.$$

Assume that $\mathfrak{B}$ is infinite. Then there are a $\mu \in \mathcal{N}$, a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}$ and a sequence $(B_n)_{n\in\mathbb{N}}$ such that $A_m \neq A_n$ whenever $m, n \in \mathbb{N}$ are distinct and such that $B_n \subset A_n$ and

$$|\widetilde{\mu}(B_n)| > \varepsilon$$

for every $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, put

$$C_n := B_n \setminus \bigcup_{k=1}^{n-1} B_k.$$

Then

$$|\widetilde{\mu}(C_n)| = |\widetilde{\mu}(B_n)| > \varepsilon,$$

(since $\widetilde{\mu}$ vanishes on the finite subsets of $\mathbb{N}$), and so

$$|\mu(C_n)| = |\widetilde{\mu}(C_n) + \sum_{k\in C_n} \mu(\{k\})| > \varepsilon - \sum_{k\in C_n} |\mu(\{k\})|.$$

Since $(C_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets,

$$\lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \sum_{k\in C_n} |\mu(\{k\})| = 0$$

(Proposition 1.1.6.14 d $\Rightarrow$ b) and we deduce the contradiction

$$0 = \lim_{n\to\infty} |\mu(C_n)| \geq \lim_{n\to\infty} \left(\varepsilon - \sum_{k\in C_n} |\mu(\{k\})| \right) = \varepsilon .$$

Hence $\mathfrak{B}$ is finite.

It follows that the set of those all $A \in \mathfrak{A}$ for which there are $\mu \in \mathcal{M}$ and $B \subset A$ with

$$\tilde{\mu}(B) \neq 0$$

is countable. Since $\mathfrak{A}$ is uncountable, there is an $M \in \mathfrak{A}$ such that

$$\tilde{\mu}(B) = 0$$

for every $\mu \in \mathcal{M}$ and $B \subset M$. It follows that $\mu|\mathfrak{P}(M)$ is a measure for every $\mu \in \mathcal{M}$. ■

Lemma 4.3.2.14 (Nikodym's Boundedness Theorem, 1931) *Let $\mathfrak{R}$ a σ-algebra of subsets of the set T and $\mathcal{M}$ a set of bounded complex valued functions on $\mathfrak{R}$ such that*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $\mu \in \mathcal{M}$ and A, B are disjoint sets of $\mathfrak{R}$. If $\{\mu(A) \mid \mu \in \mathcal{M}\}$ is bounded for each $\mu \in \mathcal{M}$, then $\{\mu(A) \mid \mu \in \mathcal{M}, A \in \mathfrak{R}\}$ is bounded.

Assume that $\{\mu(A) \mid \mu \in \mathcal{M}, A \in \mathfrak{R}\}$ is not bounded. We use induction to construct a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ and a disjoint sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{R}$ such that for each $n \in \mathbb{N}$:

1) $|\mu_n(A_n)| > n + \sum_{k=1}^{n-1} \sup_{\mu\in\mathcal{M}} |\mu(A_k)|$,

2) $\sup\{|\mu(A)| \mid \mu \in \mathcal{M}, A \in \mathfrak{R}, A \cap \left(\bigcup_{k=1}^{n} A_k \right) = \emptyset\} = \infty$.

Take $n \in \mathbb{N}$ and assume the sequences have been constructed up to the $(n-1)$st term. By 2), there are $\mu_n \in \mathcal{M}$ and $B \in \mathfrak{R}$ such that

$$B \cap \left(\bigcup_{k=1}^{n-1} A_k \right) = \emptyset$$

and

$$|\mu_n(B)| > n + \sum_{k=1}^{n-1} \sup_{\mu\in\mathcal{M}} |\mu(A_k)| + \sup_{\mu\in\mathcal{M}} \left|\mu\left(T\backslash \bigcup_{k=1}^{n-1} A_k\right)\right| .$$

If

$$\sup\left\{|\mu(A)| \mid \mu\in\mathcal{M},\ A\in\mathfrak{R},\ A\cap\left(\left(\bigcup_{k=1}^{n-1} A_k\right)\cup B\right)=\emptyset\right\}=\infty,$$

then define

$$A_n := B .$$

Then 1) and 2) are trivially fulfilled. Otherwise, put

$$A_n := T\backslash\left(\left(\bigcup_{k=1}^{n-1} A_k\right)\cup B\right) .$$

Then 2) is fulfilled and

$$|\mu_n(A_n)| = \left|\mu_n\left(T\backslash\bigcup_{k=1}^{n-1} A_k\right) - \mu_n(B)\right| \geq$$

$$\geq |\mu_n(B)| - \left|\mu_n\left(T\backslash\bigcup_{k=1}^{n-1} A_k\right)\right| \geq$$

$$\geq |\mu_n(B)| - \sup_{\mu\in\mathcal{M}}\left|\mu\left(T\backslash\bigcup_{k=1}^{n-1} A_k\right)\right| > n + \sum_{k=1}^{n-1}\sup_{\mu\in\mathcal{M}}|\mu(A_k)| .$$

This completes the inductive construction.

Note that

$$\sum_{n\in\mathbb{N}} |\mu(A_n)| < \infty$$

whenever $\mu\in\mathcal{M}$ (Proposition 1.1.6.14 d $\Rightarrow$ b). Given $n\in\mathbb{N}$, define

$$\nu_n : \mathfrak{P}(\mathbb{N}) \longrightarrow \mathbb{C},\quad A \longmapsto \mu_n\left(\bigcup_{k\in A} A_k\right) .$$

Then ν_n is bounded and

$$\nu_n(A\cup B) = \nu_n(A) + \nu_n(B)$$

for every $n \in \mathbb{N}$ and disjoint subsets A, B of $\mathbb{N}$. By Lemma 4.3.2.13, there is an infinite subset M of $\mathbb{N}$ such that $\nu_n|\mathfrak{P}(M)$ is a measure whenever $n \in \mathbb{N}$. There is a strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ in M such that

$$\sum_{k=p_{n+1}}^{\infty} |\nu_{p_n}(\{k\})| < 1$$

whenever $n \in \mathbb{N}$. Thus, by 1),

$$\left|\mu_{p_n}\left(\bigcup_{k\in\mathbb{N}} A_{p_k}\right)\right| \geq |\mu_{p_n}(A_{p_n})| - \sum_{k=1}^{n-1} |\mu_{p_n}(A_{p_n})| - \left|\mu_{p_n}\left(\bigcup_{k=n+1}^{\infty} A_{p_k}\right)\right| \geq$$

$$\geq p_n - |\nu_{p_n}(\{p_k \mid k \in \mathbb{N},\, k > n\})| =$$

$$= p_n - \left|\sum_{k=n+1}^{\infty} \nu_{p_n}(\{p_k\})\right| \geq p_n - \sum_{k=n+1}^{\infty} |\nu_{p_n}(\{p_k\})| > p_n - 1\,.$$

Hence

$$\sup_{\mu\in\mathcal{M}} \left|\mu\left(\bigcup_{k\in\mathbb{N}} A_{p_k}\right)\right| \geq \sup_{n\in\mathbb{N}} \left|\mu_{p_n}\left(\bigcup_{k\in\mathbb{N}} A_{p_k}\right)\right| = \infty$$

and this is a contradiction. ■

Theorem 4.3.2.15 *Let $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). Let E be unital and $\mathbb{C}$-order σ-complete. If A is a subset of E' (of $\operatorname{Re} E'$) for which $\{x'(p) \mid x' \in A\}$ is bounded whenever $p \in \operatorname{Pr} E$, then A is bounded. In particular, if $(x'_n)_{n\in\mathbb{N}}$ is a sequence in E' such that for each $p \in \operatorname{Pr} E$, $(x'_n(p))_{n\in\mathbb{N}}$ converges, then there is an $x' \in E'$ such that*

$$\lim_{n\to\infty} x'_n(x) = x'(x) \qquad \left(\lim_{n\to\infty} x'_n(x) = x'(\operatorname{re} x)\right)$$

whenever $x \in E$.

By Banach–Steinhaus Theorem (Theorem 1.4.1.2), it is sufficient to prove that $\{x'(x) \mid x' \in A\}$ is bounded for every $x \in E$. Hence by Theorem 4.2.2.9 a), it suffices to show that $\{x'(x) \mid x' \in A\}$ is bounded whenever $x \in E_+^{\#}$.

So take $x \in E_+^{\#}$ and let $\mathfrak{B}$ be the σ-algebra of Borel sets of $\sigma(x)$. By hypothesis (and Corollary 4.3.2.5 a),b)) $\{x'(e_B(x)) \mid x' \in A\}$ is bounded whenever $B \in \mathfrak{B}$. By Nikodym's Boundedness Theorem (Lemma 4.3.2.14),

$$\alpha := \sup\{|x'(e_B(x))| \mid x' \in A,\, B \in \mathfrak{B}\} < \infty\,.$$

Given $n \in \mathbb{N}$, define

$$A_n := \sigma(x) \cap \bigcup_{k \in \mathbb{N}} \left[\frac{2k-1}{2^n}, \frac{2k}{2^n} \right[.$$

By Corollary 4.3.2.11,

$$\left(\sum_{k=1}^{n} \frac{1}{2^k} e_{A_k}(x) \right)_{n \in \mathbb{N}}$$

converges to x. Since

$$\left| x' \left(\sum_{k=1}^{n} \frac{1}{2^k} e_{A_k}(x) \right) \right| \leq \sum_{k=1}^{n} \frac{1}{2^k} |x'(e_{A_k}(x))| \leq \alpha$$

whenever $x' \in A$ and $n \in \mathbb{N}$, it follows

$$|x'(x)| = \lim_{n \to \infty} \left| x' \left(\sum_{k=1}^{n} \frac{1}{2^k} e_{A_k}(x) \right) \right| \leq \alpha$$

whenever $x' \in A$ and therefore

$$\sup_{x' \in A} |x'(x)| < \infty .$$

We now prove the last assertion. By the first assertion, $(x'_n)_{n \in \mathbb{N}}$ is equicontinuous. Since the vector subspace of E generated by $\Pr E$ is dense in E (in $\operatorname{Re} E$) (Corollary 4.3.2.4) $(x'_n(x))_{n \in \mathbb{N}}$ converges for every $x \in E$ $(x \in \operatorname{Re} E)$. Define

$$x' : E \longrightarrow \mathbb{C}, \quad x \longmapsto \lim_{n \to \infty} x'_n(x) \quad (x' : E \longrightarrow \mathbb{R}, \quad x \longmapsto \lim_{n \to \infty} x'(\operatorname{re} x)) .$$

Thus x' has the required properties (Proposition 1.2.1.7). ■

Definition 4.3.2.16 (0) *Let T be a set, $\mathfrak{T}$ a σ-algebra on T, and E a unital C^*-algebra. An E-**valued spectral measure on** $\mathfrak{T}$ is a map*

$$\mu : \mathfrak{T} \longrightarrow \Pr E$$

such that:

a) $A, B \in \mathfrak{T}$, $A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$, $\mu(A)\mu(B) = 0$.

b) $\mu(T) = 1$.

c) *if $((A_n)_{n \in \mathbb{N}}$ is a decreasing sequence in $\mathfrak{T}$ with empty intersection, then 0 is the infimum of $(\mu(A_n))_{n \in \mathbb{N}}$ in E.*

If μ is a spectral measure on $\mathfrak{T}$, then $x' \circ \mu$ is a measure on $\mathfrak{T}$ for every $x' \in E^\sigma$.

Proposition 4.3.2.17 (0) *Let T be a set, $\mathfrak{T}$ a σ–algebra on T,*

$$\mathcal{B} := \{x \in \ell^\infty(T) \mid x \text{ is } \mathfrak{T}\text{–measurable}\},$$

E a unital C^*–algebra, and μ an E–valued spectral measure on $\mathfrak{T}$. Then there is a unique continuous linear map

$$\mathcal{B} \longrightarrow E, \quad x \longmapsto \int x d\mu$$

such that

$$\int e_A d\mu = \mu(A)$$

for every $A \in \mathfrak{T}$. This map is a homomorphism of involutive unital algebras and

$$\bigwedge_{n\in\mathbb{N}} \int x_n d\mu = \int \left(\bigwedge_{n\in\mathbb{N}} x_n \right) d\mu$$

for every decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_+$. If $x \in \mathcal{B}$, then

$$\sigma(\int x d\mu) \subset \sigma_{\mathcal{B}}(x) \subset \overline{x(T)}$$

and

$$f(\int x d\mu) = \int (f \circ x) d\mu$$

for any bounded Borel function on $\overline{x(T)}$.

Let $\mathcal{F}$ be the set of step functions on T with respect to $\mathfrak{T}$. $\mathcal{F}$ is a dense involutive subalgebra of $\mathcal{B}$ and there is a unique linear map

$$u : \mathcal{F} \longrightarrow E, \quad x \longmapsto \int x d\mu,$$

such that

$$\int e_A d\mu = \mu(A)$$

for every $A \in \mathfrak{T}$.

Since the unital C^*-subalgebra of E generated by $\mu(\mathfrak{T})$ is a Gelfand C^*-algebra (Corollary 4.1.2.3), we may assume that E is commutative. Take $x \in \mathcal{F}$. There is a finite family $((\alpha_\iota, A_\iota))_{\iota \in I}$ in $x(T) \times \mathfrak{T}$ such that

$$\iota, \lambda \in \mathfrak{T}\,,\ \iota \neq \lambda \Longrightarrow A_\iota \cap A_\lambda = \emptyset\,,$$

$$x = \sum_{\iota \in I} \alpha_\iota e_{A_\iota}\,.$$

Now

$$\|ux\| = \|\sum_{\iota \in I} \alpha_\iota e_{A_\iota}\| \leq \sup_{\iota \in I} |\alpha_\iota| = \|x\|$$

as can be seen using the Gelfand transform. Hence u is continuous and it can be extended uniquely to a continuous linear map

$$v : \mathcal{B} \longrightarrow E\,, \quad x \longmapsto \int x d\mu\,.$$

Take $x, y \in \mathcal{F}$. There is a finite family $((\alpha_\iota, \beta_\iota, A_\iota))_{\iota \in I}$ in $\mathbb{K} \times \mathbb{K} \times \mathfrak{T}$ such that

$$\iota, \lambda \in I\,,\ \iota \neq \lambda \Longrightarrow A_\iota \cap A_\lambda = \emptyset\,,$$

$$x = \sum_{\iota \in I} \alpha_\iota e_{A_\iota}\,, \quad y = \sum_{\iota \in I} \beta_\iota e_{A_\iota}\,.$$

Then

$$(ux)^* = \left(\sum_{\iota \in I} \alpha_\iota \mu(A_\iota)\right)^* = \sum_{\iota \in I} \overline{\alpha}_\iota \mu(A_\iota) = u\left(\sum_{\iota \in I} \overline{\alpha}_\iota e_{A_\iota}\right) = ux^*\,,$$

$$(ux)(uy) = \left(\sum_{\iota \in I} \alpha_\iota \mu(A_\iota)\right)\left(\sum_{\lambda \in I} \beta_\lambda \mu(A_\lambda)\right) =$$

$$= \sum_{\iota, \lambda \in I} \alpha_\iota \beta_\lambda \mu(A_\iota)\mu(A_\lambda) = \sum_{\iota \in I} \alpha_\iota \beta_\iota \mu(A_\iota) = u\left(\sum_{\iota \in I} \alpha_\iota \beta_\iota e_{A_\iota}\right) = u(xy)\,,$$

$$u(e_T) = \mu(T) = 1\,.$$

Hence u is a homomorphism of involutive unital algebras. By continuity, v is an involutive unital algebra homomorphism.

To prove that

$$\bigwedge_{n\in\mathbb{N}} \int x_n d\mu = \int \left(\bigwedge_{n\in\mathbb{N}} x_n \right) d\mu$$

we may assume that

$$\bigwedge_{n\in\mathbb{N}} x_n = 0 .$$

Take $\varepsilon > 0$. Given $n \in \mathbb{N}$, put

$$A_n := \{x_n > \varepsilon\} .$$

Then

$$0 \leq \int x_n d\mu \leq \int (\varepsilon e_T + \|x_1\| e_{A_n})\, d\mu = \varepsilon 1 + \|x_1\| \mu(A_n)$$

for every $n \in \mathbb{N}$ (Corollary 4.2.1.4). Since $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathfrak{T}$ with empty intersection,

$$\bigwedge_{n\in\mathbb{N}} \|x_1\| \mu(A_n) = 0 .$$

Any lower bound of $\left(\int x_n d\mu\right) n \in \mathbb{N}$ is therefore smaller than $\varepsilon 1$. Hence it must be negative (Corollary 4.2.2.2). Thus

$$\bigwedge_{n\in\mathbb{N}} \int x_n d\mu = 0 .$$

We now prove the last assertion. The relation

$$\sigma\left(\int x d\mu\right) \subset \sigma_{\mathcal{B}}(x) \subset \overline{x(T)}$$

is easy to see. Since the map

$$\mathcal{B} \longrightarrow E , \quad y \longmapsto \int y d\mu$$

is a homomorphism of involutive unital algebras

$$P\left(\int x d\mu, \left(\int x d\mu\right)^*\right) = \int P(x, \overline{x}) d\mu$$

for every $P \in \mathbb{K}[s,t]$. By the Weierstrass–Stone Theorem,

$$f\left(\int x d\mu\right) = \int (f \circ x) d\mu$$

for every $f \in \mathcal{C}(\sigma_{\mathcal{B}}(x))$. Using the property of the integral proved above, we may extend this relation to any bounded Borel function f on $\sigma_{\mathcal{B}}(x)$. ■

Proposition 4.3.2.18 (0) *Let E be a unital C^*-algebra. Take $x \in \operatorname{Sn} E$. Let $j : \sigma(x) \to \mathbb{K}$ be the inclusion map, $\mathfrak{B}$ the σ-algebra of Borel sets of $\sigma(x)$, and*

$$\mathcal{B} := \{f \in \ell^\infty(\sigma(x)) \mid f \text{ is a Borel function}\}.$$

Then any two E-valued spectral measures μ, ν on $\mathfrak{B}$ coincide whenever

$$\int j d\mu = \int j d\nu .$$

Since the maps

$$\mathcal{B} \longrightarrow E, \quad f \longmapsto \int f d\mu ,$$

$$\mathcal{B} \longrightarrow E, \quad f \longmapsto \int f d\nu ,$$

are homomorphisms of involutive algebras (Proposition 4.3.2.17),

$$\int f d\mu = f(x) = \int f d\nu$$

for all $f \in \mathcal{C}(\sigma(x))$ (Theorem 4.1.3.1 a)). Put

$$\mathfrak{R} := \{A \in \mathfrak{B} \mid \mu(A) = \nu(A)\}.$$

Furthermore, let K be a compact set of $\sigma(x)$. There is a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}(\sigma(x))_+$ such that

$$\bigwedge_{n\in\mathbb{N}}^{\mathcal{B}} f_n = e_K .$$

Thus

$$\mu(K) = \int e_K d\mu = \bigwedge_{n\in\mathbb{N}} \int f_n d\mu = \bigwedge_{n\in\mathbb{N}} \int f_n d\nu = \int e_k d\nu = \nu(K)$$

(Proposition 4.3.2.17). Hence $\mathfrak{R}$ contains the compact sets of $\mathfrak{B}$. Since the union and the intersection of every monotone sequence in $\mathfrak{R}$ belong to $\mathfrak{R}$, we see that

$$\mathfrak{R} = \mathfrak{B} .$$ ■

Theorem 4.3.2.19 (0) *Let E be $\mathbb{C}$-order σ-complete and unital. Take $x \in \operatorname{Sn} E$. Let $j : \sigma(x) \to \mathbb{K}$ be the inclusion map, $\mathfrak{B}$ the σ-algebra of Borel sets of $\sigma(x)$, and $\mathcal{B}$ the C^*-algebra of bounded Borel functions on $\sigma(x)$.*

a) *There is a unique E-valued spectral measure μ on $\mathfrak{B}$ (called* ***the spectral measure of*** *x) such that*

$$\int j d\mu = x \qquad (\textbf{\textit{the spectral decomposition of}}\, x)\,.$$

b) *For each $f \in \mathcal{B}$,*

$$\int f d\mu = f(x)$$

(Corollary 4.3.2.5 b)) and $\mu \circ \overset{-1}{f}$ is the spectral measure of $f(x)$.

c) $\mu(\mathfrak{B}) \subset \mu(\mathfrak{B})^c = \{x\}^c$.

d) *If U is a nonempty open set of $\sigma(x)$, then $\mu(U) \neq 0$.*

e) *For each $f \in \mathcal{B}$ and $\alpha \in \mathbb{K}$, $\alpha \in \sigma(f(x))$ iff $\mu\left(\overset{-1}{f}\left(U_\varepsilon^{\mathbb{K}}(\alpha)\right)\right) \neq 0$ for every $\varepsilon > 0$.*

a) The uniqueness follows from Proposition 4.3.2.18. We set

$$\mu : \mathfrak{B} \longrightarrow \Pr E\,, \quad A \longmapsto e_A(x)\,.$$

By Corollary 4.3.2.5 a), μ is a spectral measure and by Proposition 4.3.2.17, the map

$$\mathcal{B} \longrightarrow E\,, \quad f \longmapsto \int f d\mu$$

is a homomorphism of involutive unital algebras. Hence

$$\bigwedge_{n\in\mathbb{N}} \int f_n d\mu = \int \left(\bigwedge_{n\in\mathbb{N}} f_n\right) d\mu$$

for any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_+$. Take $\varepsilon > 0$. There is a finite family $((\alpha_\iota, A_\iota))_{\iota\in I}$ in $\sigma(x) \times \mathfrak{B}$ such that

$$\iota\,, \lambda \in I\,, \iota \neq \lambda \Longrightarrow A_\iota \cap A_\lambda = \emptyset\,,$$

$$\left\| j - \sum_{\iota\in I} \alpha_\iota e_{A_\iota} \right\| < \frac{\varepsilon}{2}\,.$$

Then

$$\|x - \int j d\mu\| = \left\| j(x) - \left(\sum_{\iota \in I} \alpha_\iota e_{A_\iota} \right)(x) \right\| + \left\| \int j d\mu - \int \left(\sum_{\iota \in I} \alpha_\iota e_{A_\iota} \right) d\mu \right\| =$$

$$= \left\| \left(j - \sum_{\iota \in I} \alpha_\iota e_{A_\iota} \right)(x) \right\| + \left\| \int \left(j - \sum_{\iota \in I} \alpha_\iota e_{A_\iota} \right) d\mu \right\| \leq 2 \left\| j - \sum_{\iota \in I} \alpha_\iota e_{A_\iota} \right\| < \varepsilon .$$

Hence

$$\int j d\mu = x .$$

b) The relation

$$\int f d\mu = f(x)$$

follows from a) and Corollary 4.3.2.5 a). The other assertion follows from

$$e_A(f(x)) = e_A \circ f(x) = e_{\overset{-1}{f}(A)}(x) = \mu\left(\overset{-1}{f}(A) \right) ,$$

where A is a Borel set (Corollary 4.3.2.5 e)).

c) By Corollary 4.3.2.5 c),

$$\{x\}^c \subset \mu(\mathfrak{B})^c .$$

By Corollary 4.1.4.2 b) and Corollary 4.2.2.19, $\mu(\mathfrak{B})^{cc}$ is an order faithful C^*-subalgebra of E. Since it contains $\mu(\mathfrak{B})$, it contains $f(x)$ whenever $f \in \mathcal{B}$. In particular,

$$\{x\} \subset \mu(B)^{cc} ,$$

so

$$\mu(\mathfrak{B}) \subset \mu(\mathfrak{B})^c = \mu(\mathfrak{B})^{ccc} \subset \{x\}^c$$

(Proposition 2.1.1.17 b),d),e)), and

$$\mu(\mathfrak{B}) \subset \mu(\mathfrak{B})^c = \{x\}^c .$$

d) There is an $f \in C(\sigma(x))_+$ such that $f \leq e^U$ and $\|f\| = 1$. Thus

$$0 < f(x) \leq \mu(U) .$$

e) By b), $\mu \circ \overset{-1}{f}$ is the spectral measure of $f(x)$. If $\alpha \in \sigma(f(x))$, then by d),

$$\mu\left(\overset{-1}{f}\left(U_\varepsilon^{\mathbb{K}}(\alpha)\right)\right) = \mu \circ \overset{-1}{f}\left(U_\varepsilon^{\mathbb{K}}(\alpha)\right) \neq 0$$

for every $\varepsilon > 0$. If $\alpha \notin \sigma(f(x))$, then there is an $\varepsilon > 0$ such that

$$U_\varepsilon^{\mathbb{K}}(\alpha) \cap \sigma(f(x)) = \emptyset .$$

Hence

$$\mu\left(\overset{-1}{f}\left(U_\varepsilon^{\mathbb{K}}(\alpha)\right)\right) = \mu \circ \overset{-1}{f}\left(U_\varepsilon^{\mathbb{K}}(\alpha)\right) = 0 . \qquad \blacksquare$$

Definition 4.3.2.20 (0) *Let E be $\mathbb{C}$-order σ-complete and unital. Take $x \in \operatorname{Sn} E$ and let μ be the spectral measure of x. An* ***atom*** *of x is an $\alpha \in \sigma(x)$, such that $\mu(\{\alpha\}) \neq 0$. x is called* ***atomless*** *if it has no atoms. x is called* ***atomic*** *if for every Borel set A of $\sigma(x)$, $\mu(A)$ is the supremum of $(\mu(\{\alpha\}))_{\alpha \in A}$.*

Corollary 4.3.2.21 *Let E be $\mathbb{C}$-order σ-complete and unital. Take $x \in \operatorname{Sn} E$ and let f be a bounded Borel function on $\sigma(x)$. If x is atomic, then $f(x)$ is atomic too.*

Let μ be the spectral measure of x. By Theorem 4.3.2.19 b), $\mu \circ \overset{-1}{f}$ is the spectral measure of $f(x)$. Let A be a Borel set of $\mathbb{K}$. Since x is atomic, $\mu(\overset{-1}{f}(A))$ is the supremum of $(\mu(\{\alpha\}))_{\alpha \in \overset{-1}{f}(A)}$ and $\mu(\overset{-1}{f}(\beta))$ is the supremum of $(\mu(\{\alpha\}))_{\alpha \in \overset{-1}{f}(\beta)}$ for any $\beta \in \mathbb{K}$. Hence $\mu \circ \overset{-1}{f}(A)$ is the supremum of $(\mu \circ \overset{-1}{f}(\{\beta\}))_{\beta \in A}$ and $f(x)$ is atomic. $\blacksquare$

Proposition 4.3.2.22 *Let E be $\mathbb{C}$-order complete and unital. Take $x \in \operatorname{Sn} E$ and let μ be the spectral measure of x. Then x is atomic iff 1 is the supremum of $(\overset{-1}{\mu}(\{\alpha\}))_{\alpha \in \sigma(x)}$.*

Assume that 1 is the supremum of $(\overset{-1}{\mu}(\{\alpha\}))_{\alpha \in \sigma(x)}$. Let A be a Borel set of $\sigma(x)$ and let y, z be the suprema of $(\overset{-1}{\mu}(\{\alpha\}))_{\alpha \in A}$ and $(\overset{-1}{\mu}(\{\alpha\}))_{\alpha \in \sigma(x) \setminus A}$, respectively (Corollary 4.3.2.6). Then

$$y \leq \mu(A), \quad z \leq \mu(\sigma(x) \setminus A)$$

and

$$1 = y + z \leq \mu(A) + \mu(\sigma(x) \setminus A) = \mu(\sigma(x)) = 1 .$$

Hence

$$\mu(A) = y$$

and x is atomic.

The reverse implication is trivial. $\blacksquare$

4.3.3 The Carrier

Definition 4.3.3.1 (0) *Let F be an E-module. Take $a \in F$ and put*

$$G := \{x \in E \mid ax = 0\}, \quad (G := \{x \in E \mid xa = 0\}).$$

*A **right (left) carrier of** a is an orthogonal projection p of E such that*

$$ap = a, pG = \{0\}, \quad (pa = a, Gp = \{0\}).$$

*A **carrier of** a is an orthogonal projection of E, which is at the same time a right and a left carrier of a.*

0 is a right (left) carrier of a iff $a = 0$.

Let T be a locally compact space. Put $E := \mathcal{C}_0(T)$ and $F := \mathcal{C}_0(T)$ (resp. $F := \mathcal{M}_b(T)$). An element $a \in F$ has a right carrier iff $\overline{\{a \neq 0\}}$ (if $\operatorname{Supp} a$) is relatively compact and open, in which case the characteristic function of this set is the carrier of a.

Proposition 4.3.3.2 (0) *Let F be an E-module. Take $a \in F$, $p \in \operatorname{Pr} E$ and put*

$$A := \{q \in \operatorname{Pr} E \mid aq = a \quad (resp.\ qa = a)\}.$$

a) $q \in A$, $r \in \operatorname{Pr} E$, $q \leq r \Rightarrow r \in A$.

b) *If p is a right (left) carrier of a, then it is the smallest element of A.*

c) *There is at most one right (left) carrier of a.*

d) *If $x, y \in \operatorname{Re} E$ such that $|x| \leq |y|$, and if p, q are the right carriers of x and y, respectively, then $p \leq q$.*

a) We have

$$ar = aqr = aq = a \quad (ra = rqa = qa = a)$$

(Corollary 4.2.7.6 a $\Rightarrow$ c (resp. a $\Rightarrow$ d)), so $r \in A$.

b) If $q \in A$, then

$$a(q - p) = aq - ap = a - a = 0 \quad ((q - p)a = qa - pa = a - a = 0),$$

$$0 = p(q - p) = pq - p \quad (0 = (q - p)p = qp - p),$$

$$pq = p \quad (qp = p)$$

$$p \leq q\,,$$

by Corollary 4.2.7.6 c $\Rightarrow$ a (resp. d $\Rightarrow$ a).

c) follows from b).

d) We may assume $y \in E^{\#}$. By Proposition 4.2.7.1 f $\Rightarrow$ a, $|y| \leq q$, so that $|x| \leq q$. By Proposition 4.2.7.1 a $\Rightarrow$ f, $xq = x$. By b), $p \leq q$. ■

Proposition 4.3.3.3 (0) *Let F be an involutive E-module. Take $a \in F$ and let p be the right (left) carrier of a. Then p is the left (right) carrier of a^*.*

We have

$$pa^* = (ap)^* = a^* \quad (a^*p = (pa)^* = a^*)\,.$$

Take $x \in E$ with

$$xa^* = 0 \quad (a^*x = 0)\,.$$

Then

$$ax^* = 0 \quad (x^*a = 0)\,,$$

so that

$$px^* = 0 \quad (x^*p = 0)\,,$$

$$xp = 0 \quad (px = 0)\,.$$

Hence p is the left (right) carrier of a^*. ■

Remark. This proposition allows us to deduce the properties of the left carrier from those of the right carrier.

Proposition 4.3.3.4 *Let F be an E-module. Take $a \in F$. Let p be the right carrier of a. Take $x \in E$ with*

$$ax = xa\,.$$

a) $px = pxp$.

b) *If F is an involutive E-module and $a \in \operatorname{Re} F$, then*

$$px = xp\,.$$

c) *If $F = E$ and $a \in \operatorname{Re} E$, then*

$$\{a\}^c \subset \{p\}^c\,.$$

a) We have

$$a(xp - px) = axp - apx = xap - ax = xa - ax = 0\,,$$

so that

$$0 = p(xp - px) = pxp - px\,,$$

$$px = pxp\,.$$

b) and c). We have

$$ax^* = x^*a\,.$$

Hence by a),

$$px^* = px^*p\,.$$

Thus

$$(xp)^* = px^* = px^*p = (pxp)^* = (px)^*\,,$$

$$xp = px\,. \qquad \blacksquare$$

Proposition 4.3.3.5 (0) *Let E be unital and F a unital E-module. Take $a \in F$ and let p be the right carrier of a.*

a) $\{x \in E \mid ax = 0\} = (1 - p)E\,.$

b) *If $F = E$ and a is invertible then $p = 1$.*

a) Take $x \in E$ with

$$ax = 0\,.$$

Then

$$x = (1-p)x + px = (1-p)x \in (1-p)E\,.$$

Hence

$$\{x \in E \mid ax = 0\} \subset (1-p)E\,.$$

Take $y \in E$. Then

$$a(1-p)y = (a-ap)y = 0\,,$$

so that

$$(1-p)E \subset \{x \in E \mid ax = 0\}\,.$$

b) follows from

$$p = a^{-1}ap = a^{-1}a = 1\,. \qquad \blacksquare$$

Proposition 4.3.3.6 *Suppose E is unital and $\mathbb{C}$-order complete. Let F be a unital E-module. Then every element of $\dot{F}$ has both a right carrier and a left carrier.*

Take $a \in \dot{F}$. Put

$$A := \{p \in \Pr E \mid ap = a\}\,.$$

Then $1 \in A$. Take $p, q \in A$. Then $((pqp)^n)_{n\in\mathbb{N}}$ is a decreasing sequence in E (Corollary 4.2.2.5) and its infimum r in E_+ belongs to $\Pr E$ and is the infimum of p and q in E_+ (Corollary 4.2.7.13). We have

$$ar = \lim_{n\to\infty} a(pqp)^n = a\,.$$

Hence $r \in A$ and A is downward directed.

Let $\mathfrak{F}$ be the lower section filter of A and p the infimum of A in E_+. By Corollary 4.2.7.10, $p \in \Pr E$. Thus

$$ap = \lim_{q,\mathfrak{F}} aq = a\,.$$

Hence p is the smallest element of A.

Take $x \in E^{\#}$ with

$$ax = 0\,.$$

Then

$$a(xx^*) = 0 = 0a\,.$$

By the Fuglede–Putnam Theorem

$$a(xx^*)^{\frac{1}{n}} = 0^{\frac{1}{n}}a = 0$$

for every $n \in \mathbb{N}$. $((xx^*)^{\frac{1}{n}})_{n\in\mathbb{N}}$ is an increasing sequence in $E_+^{\#}$ (Corollary 4.2.7.2). Let q be its supremum. By Proposition 4.2.4.2 f), $q \in \Pr E$. Thus

$$aq = \lim_{n\to\infty} a(xx^*)^{\frac{1}{n}} = 0\,,$$

so that

$$a(1-q) = a - aq = a\,,$$

$$1 - q \in A\,,$$

$$p \leq 1 - q\,,$$

$$xx^* \leq q \leq 1 - p\,,$$

$$xx^*(1-p) = xx^*$$

(Proposition 4.2.7.1 d $\Rightarrow$ f),

$$xx^*p = 0\,,$$

$$x^*p = 0$$

(Remark a) of Definition 4.1.1.1),

$$px = 0\,.$$

Hence p is the right carrier of a.

The proof of the existence of a left carrier of a is similar. ■

Proposition 4.3.3.7 (0) *Take $x \in E$. If p is the right carrier of x, then $p \in \{x^*x\}^{cc}$.*

Take $y \in \{x^*x\}^c$. Then

$$x^*x(py - yp) = x^*xpy - x^*xyp = x^*xy - yx^*xp = yx^*x - yx^*x = 0\,,$$

$$x(py - yp) = 0$$

(Remark a) of Definition 4.1.1.1),

$$0 = p(py - yp) = py - pyp\,,$$

$$py = pyp\,.$$

Since $y^* \in \{x^*x\}^c$, we deduce that

$$(yp)^* = py^* = py^*p = (pyp)^*\,,$$

$$yp = pyp = py\,,$$

$$p \in \{x^*x\}^{cc}\,. \qquad \blacksquare$$

Theorem 4.3.3.8 (0) *Take $(x,p) \in E \times \Pr E$. Then the following are equivalent:*

a) *p is the right carrier of x.*

b) *(resp. c)) $p \in \{x^*x\}^{cc}$ and if F is a (maximal) Gelfand C^*-subalgebra of E containing $\{x^*x, p\}$, then*

$$\overset{F}{\widehat{p}} = e_{\overline{A}}^{\sigma(F)}\,,$$

where

$$A := \{\overset{F}{\widehat{x^*x}} \neq 0\}\,.$$

a $\Rightarrow$ b. By Proposition 4.3.3.7, $p \in \{x^*x\}^{cc}$. It is obvious that $\overset{F}{\widehat{p}} = 1$ on $\overline{A}$. Take $f \in \mathcal{C}_0(\sigma(F))$ with $f = 0$ on $\overline{A}$. Then there is a $y \in F$ with

$$\overset{F}{\widehat{y}} = f\,.$$

We see successively that

$$\overset{F}{\widehat{x^*xy}} = \overset{F}{\widehat{x^*x}} f = 0\,,$$

$$x^*xy = 0\,,$$

$$xy = 0$$

(Remark a) of Definition 4.1.1.1),

$$py = 0\,,$$

$$\overset{F}{\widehat{p}} f = \overset{F}{\widehat{py}} = 0\,.$$

Since f is arbitrary, $\overset{F}{\widehat{p}} = 0$ on $\sigma(F)\backslash A$. Hence

$$\overset{F}{\widehat{p}} = e_A^{\sigma(F)}\,.$$

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. Since $\{x^*x, p\}$ is commutative, there is a maximal Gelfand C^*-subalgebra F of E containing $\{x^*x, p\}$ (Corollary 4.1.2.3). We have successively that

$$\overset{F}{\widehat{x^*xp}} = \overset{F}{\widehat{x^*x}}\,\overset{F}{\widehat{p}} = \overset{F}{\widehat{x^*x}}\,,$$

$$x^*xp = x^*x\,,$$

$$xp = x$$

(Remark c) of Definition 4.1.1.1).

Take $y \in E$ with

$$xy = 0\,.$$

Then

$$x^*xyy^* = 0\,,$$

$$yy^*x^*x = 0\,,$$

$$yy^* \in \{x^*x\}^c\,,$$

$$yy^*p = pyy^*\,.$$

Hence there is a maximal Gelfand C^*–subalgebra F of E containing $\{x^*x, yy^*, p\}$ (Corollary 4.1.2.3). We have that

$$\overset{F}{\widehat{yy^*}}\,\overset{F}{\widehat{x^*x}} = \overset{F}{\widehat{yy^*x^*x}} = 0$$

and so $\overset{F}{\widehat{yy^*}}$ vanishes on $\overline{A}$. It follows that

$$\overset{F}{\widehat{pyy^*}} = \overset{F}{\widehat{p}}\,\overset{F}{\widehat{yy^*}} = 0\,,$$

$$pyy^* = 0\,,$$

$$py = 0$$

(Remark b) of Definition 4.1.1.1). Thus p is the right carrier of x. ■

Corollary 4.3.3.9 (0) *A self–normal element of a C^*–algebra possesses a right carrier iff it possesses a left carrier and in this case the two carriers coincide.*

This follows immediately from Theorem 4.3.3.8. ■

Corollary 4.3.3.10 (0) *Given $\alpha > 0$ and $(x,p) \in E \times \Pr E$, the following are equivalent:*

a) *p is the right carrier of x.*

b) *p is the carrier of $|x|^\alpha$.*

c) *p is the left carrier of x^*.*

a $\Leftrightarrow$ c follows from Proposition 4.3.3.3.

a $\Leftrightarrow$ b. By the Remark a) of Definition 4.1.1.1,

$$xy = 0 \Longleftrightarrow x^*xy = 0$$

for every $y \in E$. It follows that p is the right carrier of x iff it is the carrier of x^*x. Take $\beta, \gamma \in \mathbb{R}_+\backslash\{0\}$. By Proposition 4.2.4.2 b),

$$E(|x|^\beta) = E(|x|^\gamma), \quad \{|x|^\beta\}^c = \{|x|^\gamma\}^c.$$

If F is a Gelfand C^*-subalgebra of E containing $|x|^\beta$ and p, then it contains $|x|^\gamma$ and

$$\{\overset{F}{\widehat{|x|^\beta}} \neq 0\} = \{\overset{F}{\widehat{|x|^\gamma}} \neq 0\}.$$

By Theorem 4.3.3.8, p is the carrier of x^*x iff it is the carrier of $|x|^\alpha$. ■

Corollary 4.3.3.11 (0) *Take $x \in E$ and let*

$$f := e^{\sigma(x^*x)}_{\sigma(x^*x)\setminus\{0\}}.$$

If f is continuous or if E is unital and $\mathbb{C}$-order σ-complete, then $f(x^*x)$ (Corollary 4.3.2.5 b)) is the right carrier of x.

We have

$$f(x^*x) \in \Pr E$$

and

$$\{x^*x\}^c \subset \{f(x^*x)\}^c$$

(Corollary 4.3.2.5 a),c)). Thus

$$f(x^*x) \in \{f(x^*x)\}^{cc} \subset \{x^*x\}^{cc}.$$

Let F be a maximal Gelfand C^*-subalgebra of E containing $\{x^*x\}^{cc}$ (Corollaries 4.1.2.3 and 4.1.4.2 c)). Then $f(x^*x) \in F$ and

$$\left\{\overset{F}{\widehat{f(x^*x)}} \neq f \circ \overset{F}{\widehat{x^*x}}\right\}$$

is meager (Corollary 4.3.2.5 c)). Put

$$A := \left\{\overset{F}{\widehat{x^*x}} \neq 0\right\}.$$

Then

$$f \circ \overset{F}{\widehat{x^*x}} = e^{\sigma(F)}_A.$$

It follows that

$$\overset{F}{\widehat{f(x^*x)}} = e^{\sigma(F)}_A.$$

By Theorem 4.3.3.8 c $\Rightarrow$ a, $f(x^*x)$ is the right carrier of x. ■

Corollary 4.3.3.12 (0) *Let E be unital and $\mathbb{C}$-order σ-complete, F an order σ-faithful unital C^*-subalgebra of E, and take $x \in F$. Then the left (right) carrier of x with respect to E belongs to F and is the left (right) carrier of x with respect to F.*

Define

$$f := e^{\sigma(x^*x)}_{\sigma(x^*x)\setminus\{0\}}.$$

By Corollary 4.3.3.11, $f(x^*x)$ is the right carrier with respect to E and by Corollary 4.3.2.5 c), it belongs to F. ■

Corollary 4.3.3.13 *Assume that E is unital and $\mathbb{C}$-order σ-complete. If x is the supremum of a subset A of E_+, then the carrier of x is the supremum in $\Pr E$ of the carriers of the element of A.*

We may assume that $x \in E^{\#}$. Given $y \in E^{\#}_+$, let p_y denote its carrier (Corollary 4.3.3.11). By Proposition 4.2.7.1 d $\Leftrightarrow$ e,

$$y \leq p \Longleftrightarrow py = y$$

for all $(y,p) \in E^{\#}_+ \times \Pr E$.

Given $y \in A$,

$$y \leq x \leq p_x,$$

so that

$$p_x y = y,$$

and p_x is an upper bound for $(p_y)_{y\in A}$ (Proposition 4.3.3.2 b)). Let q be an upper bound for $(p_y)_{y\in A}$ in $\Pr E$. Then

$$y \leq p_y \leq q$$

for every $y \in A$. Thus

$$x \leq q,$$

$$qx = x,$$

$$p_x \leq q$$

(Proposition 4.3.3.2 b)). Hence p_x is the supremum of $(p_y)_{y\in A}$. ■

Corollary 4.3.3.14 (0) *Let E be unital and $\mathbb{C}$-order σ-complete. Take $x \in \mathrm{Sn}\, E$. Let p be the carrier of x, f a bounded Borel function on $\sigma(x)$ such that*

$$0 \in \sigma(x) \Longrightarrow f(0) = 0\,,$$

and q the carrier of $f(x)$. Put

$$g := e^{\mathbb{K}}_{\mathbb{K}\setminus\{0\}}\,.$$

Then

$$g \circ f(x) = q \leq p = g(x)\,.$$

Let F be a maximal Gelfand C^*-subalgebra of E containing x (Corollary 4.1.2.3 d)). Then $f(x)$, $g(x)$, $g \circ f(x) \in F$ and the sets

$$\left(\overset{F}{\widehat{g(x)}} \neq g \circ \overset{F}{\widehat{x}}\right)\,, \quad \left(\overset{F}{\widehat{g \circ f(x)}} \neq g \circ f \circ \overset{F}{\widehat{x}}\right)$$

are meager (Corollary 4.3.2.5 c)). Since

$$g \circ f \circ \overset{F}{\widehat{x}} \leq g \circ \overset{F}{\widehat{x}}\,,$$

we see that

$$\overset{F}{\widehat{g \circ f(x)}} \leq \overset{F}{\widehat{g(x)}}\,,$$

$$g \circ f(x) \leq g(x)\,.$$

By Corollary 4.3.3.11,

$$q = g(f(x)^* f(x)) = g(f(x)) \leq g(x) = g(x^* x) = p\,. \qquad \blacksquare$$

Proposition 4.3.3.15 *Let F be an involutive E-module, $a, b \in \mathrm{Re}\, F$, and p, q the carriers of a and b, respectively. If $pq = 0$ then $p + q$ is the carrier of $a + b$. If $x \in \mathrm{Re}\, E$ and p, q are the carriers of x^+ and x^-, respectively, then $pq = 0$ and $p + q$ is the carrier of x.*

By Corollary 4.2.7.5, $p + q \in \mathrm{Pr}\, E$. From

$$aq = apq = 0\,, \quad bp = bqp = 0$$

it follows

$$(a+b)(p+q) = ap + bq = a + b\,.$$

Let $y \in E$ with

$$(a+b)y = 0\,.$$

Then

$$ay = p(a+b)y = 0\,.$$

It follows

$$py = 0\,,$$

$$qy = 0\,,$$

$$(p+q)y = 0\,.$$

Hence $p+q$ is the carrier of $a+b$.

In order to prove the final assertion, we remark that from

$$x^+x^- = 0$$

it follows

$$px^- = 0\,, \quad pq = 0\,.$$

By the first assertion, $p+q$ is the carrier of x. ■

Proposition 4.3.3.16 *We assume* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{K} = \mathbb{R}$ *and* $\overset{\circ}{E}$ *order* σ*-complete. Take* $x \in E^{\#}$. *If* $\left((x^*x)^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ *(resp.* $\left((xx^*)^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$*) has a supremum* p *in* E, *then* p *is the right (left) carrier of* x.

By Proposition 4.2.4.2 b),

$$\left\{(x^*x)^{\frac{1}{n}}\right\}^c = \{x^*x\}^c$$

for every $n \in \mathbb{N}$. Hence

$$p \in \{x^*x\}^{cc}$$

by Proposition 4.2.2.17. Let F be a Gelfand C^*-subalgebra of E, containing $\{x^*x, p\}$. It is easy to see that

$$\overset{F}{\widehat{p}} = e_{\overset{\sigma(F)}{A}},$$

where

$$A := \left\{ \overset{F}{\widehat{x^*x}} \neq 0 \right\}.$$

By Theorem 4.3.3.8 b $\Rightarrow$ a, p is the right carrier of x. ■

Example 4.3.3.17 *Let*

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

be a non-zero, non-invertible element of the C^*-algebra $\mathcal{L}(\mathbb{K}^2)$ (Example 4.1.1.7), i.e.

$$\alpha\delta - \beta\gamma = 0.$$

Take $\varphi \in \left[0, \frac{\pi}{2}\right]$ with

$$tg\varphi = \left|\frac{\gamma}{\alpha}\right| = \left|\frac{\delta}{\beta}\right| \qquad \left(tg\varphi = \left|\frac{\beta}{\alpha}\right| = \left|\frac{\delta}{\gamma}\right|\right)$$

and $\theta \in \mathbb{R}$ with

$$\frac{\gamma}{\alpha} = \left|\frac{\gamma}{\alpha}\right| e^{i\theta} \qquad \left(\frac{\beta}{\alpha} = \left|\frac{\beta}{\alpha}\right| e^{i\theta}\right)$$

if $\varphi \in]0, \frac{\pi}{2}[$. Then

$$\begin{bmatrix} \cos^2\varphi & e^{i\theta}\cos\varphi\sin\varphi \\ e^{-i\theta}\cos\varphi\sin\varphi & \sin^2\varphi \end{bmatrix}$$

is the left (right) carrier of

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

Use Example 4.1.2.19. ■

Example 4.3.3.18 *Take $n \in \mathbb{N}$ and let $\mathcal{L}(\mathbb{K}^n)$ be the C^*-algebra defined in Example 4.1.1.7. Given $a \in \mathcal{L}(\mathbb{K}^n)$, define*

$$\widetilde{a} : \mathcal{L}(\mathbb{K}^n) \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{i,j=1}^{n} x_{ij} a_{ji}$$

and identify $\mathcal{L}(\mathbb{K}^n)$ with $\mathcal{L}(\mathbb{K}^n)'$ using the map

$$\mathcal{L}(\mathbb{K}^n) \longrightarrow \mathcal{L}(\mathbb{K}^n)', \quad a \longmapsto \widetilde{a}$$

(Example 4.2.3.2). Then, for every $a \in \mathcal{L}(\mathbb{K}^n)$, the left (right) carrier of a and the left (right) carrier of $\widetilde{a}$ coincide.

The assertion follows immediately from Example 4.3.1.12. ■

Proposition 4.3.3.19 (0) *Take $x \in E$. If $x^*x \in \Pr E$ ($xx^* \in \Pr E$), then x^*x is the right (xx^* is the left) carrier of x.*

Now

$$xx^*x = x$$

(Corollary 4.1.2.22 a $\Rightarrow$ c, b $\Rightarrow$ c) and for each $y \in E$,

$$xy = 0 \Longrightarrow x^*xy = 0,$$

$$yx = 0 \Longrightarrow yxx^* = 0. \qquad \blacksquare$$

Theorem 4.3.3.20 (0) (Kadison, 1951) *Take $x \in E^{\#}$ and put*

$$u := x^*x, \quad v := xx^*.$$

Then the following are equivalent:

a) *x is an extreme point of $E^{\#}$.*

b) *$y \in E \Rightarrow (1-v)y(1-u) = 0$.*

c) *$(x,0)$ is an extreme point of $\overset{\circ}{E}{}^{\#}$.*

If these equivalent conditions apply, then:

d) *$u, v \in \Pr E, \quad x = xu = vx$.*

e) *u and v are the right and the left carrier of x, respectively, E is unital, and*

$$uv = vu\,, \quad 1 = u + v - uv\,.$$

f) *x is an extreme point of $E(x)$ and $1 \in E(x)$.*

We have

$$xu = xx^*x = vx\,, \quad \|u\| = \|x^*x\| = \|x\|^2 \leq 1\,.$$

a $\Rightarrow$ d. Define

$$z := u^{\frac{1}{3}}\,.$$

Then

$$\|z\| = \left\|u^{\frac{1}{3}}\right\| = \|u\|^{\frac{1}{3}} \leq 1$$

(Proposition 4.2.4.2 b)), so that

$$\sigma(z) \subset [0,1]$$

(Corollary 4.2.1.16 a $\Rightarrow$ c). By Corollary 4.2.4.14, there is some $y \in E$ with

$$x = yz\,, \quad \|y\| \leq \left\|u^{\frac{1}{2}-\frac{1}{3}}\right\| = \left\|u^{\frac{1}{6}}\right\| = \|u\|^{\frac{1}{6}} \leq 1$$

(Proposition 4.2.4.2 b)). Since

$$\|xz\| \leq \|x\|\,\|z\| \leq 1\,,$$

$xz \in E^{\#}$. Define

$$f : \sigma(z) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto 2\alpha - \alpha^2\,.$$

Then

$$\|2z - z^2\| = \|f(z)\| = \|f\| \leq 1$$

(Theorem 4.1.3.1 b)), so that

$$\|x(2\cdot 1 - z)\| = \|yz(2\cdot 1 - z)\| \leq \|y\|\,\|2z - z^2\| \leq 1\,,$$

$$x(2\cdot 1 - z) \in E^{\#}\,.$$

We have

$$x = \frac{1}{2}(x(2 \cdot 1 - z) + xz)\,.$$

Since x is an extreme point of $E^{\#}$, it follows successively that

$$x = xz\,,$$

$$u = x^*x = x^*xz = uu^{\frac{1}{3}} = u^{\frac{4}{3}}\,,$$

$$u \in \Pr E \quad \text{(Corollary 4.2.4.3 a} \Rightarrow \text{c)}\,,$$

$$v \in \Pr E\,, \quad x = xu = vx \quad \text{(Corollary 4.1.2.22)}\,.$$

a $\Rightarrow$ b. We may assume that $\|y\| \leq 1$. Put

$$z := (1 - v)y(1 - u)\,.$$

Then

$$\|z^*z\| = \|z\|^2 = \|(1 - v)y(1 - u)\|^2 \leq \|1 - v\|^2\|y\|^2\|1 - u\|^2 \leq 1$$

(Corollary 4.2.1.17 a $\Rightarrow$ c). We show that $x \pm z \in E^{\#}$.

$$\|x \pm z\|^2 = \|(x^* \pm z^*)(x \pm z)\| = \|x^*x \pm x^*z \pm z^*x + z^*z\|\,.$$

By d),

$$x^*z = x^*v(1 - v)y(1 - u) = 0\,,$$

$$z^*x = (x^*z)^* = 0\,,$$

$$x^*xz^*z = x^*xu(1 - u)y^*(1 - v)z = 0\,,$$

$$z^*zx^*x = z^*(1 - v)y(1 - u)ux^*x = 0\,,$$

so that

$$\|x \pm z\|^2 = \|x^*x + z^*z\|\,.$$

The C^*-subalgebra of E generated by $\{x^*x, z^*z\}$ being a Gelfand C^*-algebra (Corollary 4.1.2.3), it follows that

$$\|x \pm z\|^2 = \|x^*x + z^*z\| = \sup\{\|x^*x\|, \|z^*z\|\} \leq 1,$$

by the Gelfand transform (Corollary 4.1.2.5). So $x \pm z \in E^{\#}$. Since x is an extreme point of $E^{\#}$ and

$$x = \frac{1}{2}(x+z) + \frac{1}{2}(x-z),$$

we deduce that

$$x = x + z,$$

$$z = 0.$$

b $\Rightarrow$ d. We have successively that

$$0 = x^*(1-v)x(1-u) = x^*(x-vx)(1-u) =$$

$$= x^*(x-xu)(1-u) = x^*x(1-u)^2 = u(1-u)^2,$$

$$\sigma(u) \subset \{0,1\} \quad \text{(Theorem 2.1.3.4 a)},$$

$$u \in \Pr E \quad \text{(Proposition 4.1.2.21 c } \Rightarrow \text{ a)},$$

$$v \in \Pr E,\ x = xu = vx \quad \text{(Corollary 4.1.2.22)}.$$

b $\Rightarrow$ a. Take $y, z \in E^{\#}$ and $\alpha, \beta \in]0,1[$ with

$$\alpha + \beta = 1, \quad x = \alpha y + \beta z.$$

Then

$$\|y^*y\| = \|y\|^2 \leq 1, \quad \|z^*z\| = \|z\|^2 \leq 1$$

and so

$$y^*y \leq 1, \quad z^*z \leq 1$$

(Corollary 4.2.1.16 a $\Rightarrow$ b). It follows by d) that

$$u = u^3 = ux^*xu = u(\alpha y^* + \beta z^*)(\alpha y + \beta z)u =$$

$$= u(\alpha^2 y^*y + \alpha\beta(y^*z + z^*y) + \beta^2 z^*z)u \leq u(\alpha^2 y^*y + \alpha\beta(y^*y + z^*z) + \beta^2 z^*z)u \leq$$

$$\leq u(\alpha^2 1 + \alpha\beta(1+1) + \beta^2 1)u = u(\alpha+\beta)^2 1u = u^2 = u$$

(Corollary 4.2.2.4, Corollary 4.2.2.3). Thus

$$u(y^*z + z^*y)u = u(y^*y + z^*z)u\,,$$

$$\|(y-z)u\|^2 = \|u(y^*-z^*)(y-z)u\| = \|u(y^*y + z^*z - y^*z - z^*y)u\| = 0\,,$$

$$(y-z)u = 0\,.$$

A similar argument shows that

$$v(y-z) = 0\,.$$

We conclude that

$$y - z = (1-v)(y-z)(1-u) = 0\,,$$

$$x = y = z\,.$$

b $\Rightarrow$ c follows from b $\Rightarrow$ a.

c $\Rightarrow$ a is trivial.

b & d $\Rightarrow$ e. By d) and Proposition 4.3.3.19, u and v are the right and the left carrier of x, respectively.

Let $\mathfrak{F}$ be an approximate unit of E (Theorem 4.2.8.2). By b),

$$y = vy + uy - vyu$$

for every $y \in E$, so that

$$\lim_{n\to\infty} \mathfrak{F} = v + u - vu\,.$$

By Proposition 2.2.7.25 a), E is unital and

$$1 = u + v - vu\,.$$

By d),

$$uv = (vu)^* = u^* + v^* - 1^* = u + v - 1 = vu\,.$$

f) It is obvious that x is an extreme point of $E(x)$. By e), $1 \in E(x)$. ■

Corollary 4.3.3.21 (0) *If E is unital then the following are equivalent for every $x \in \mathrm{No}\,E$ ($x \in \mathrm{Re}\,E$):*

a) x *is an extreme point of* $E^\#$ *(of* $\operatorname{Re} E^\#$*).*

b) x *is an extreme point of* $E(x)^\#$ *(of* $\operatorname{Re} E(x)^\#$*).*

c) $x \in \operatorname{Un} E$.

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c. Put

$$u := x^*x = xx^*.$$

By Theorem 4.3.3.20 a $\Rightarrow$ b & d & e,

$$1 - u = (1-u)^2 = (1-u)1(1-u) = 0,$$

$$u = 1.$$

c $\Rightarrow$ a follows from Theorem 4.3.3.20 b $\Rightarrow$ a. ■

Corollary 4.3.3.22 *If* E *is unital, then the following are equivalent for every* $x \in \operatorname{Re} E$*:*

a) x *is an extreme point of* $E_+^\#$.

b) $1 - 2x$ *is an extreme point of* $\operatorname{Re} E^\#$.

c) x *is an extreme point of* $E(x,1)_+^\#$.

d) $1 - 2x \in \operatorname{Un} E$.

e) $x \in \operatorname{Pr} E$.

a $\Leftrightarrow$ b. By Corollary 4.2.1.16 a $\Leftrightarrow$ b, the map

$$\operatorname{Re} E^\# \longrightarrow E_+^\#, \quad x \longmapsto \frac{1}{2}(1-x)$$

is bijective. The map

$$E_+^\# \longrightarrow \operatorname{Re} E^\#, \quad x \longmapsto 1 - 2x$$

is its inverse.

a $\Rightarrow$ c is trivial.

c $\Rightarrow$ d. By a $\Rightarrow$ b, $1-2x$ is an extreme point of $\operatorname{Re} E(x,1)^\#$. By Corollary 4.3.3.21 a $\Rightarrow$ e, $1 - 2x \in \operatorname{Un} E$.

d $\Leftrightarrow$ e follows from Proposition 4.1.2.25.

d $\Rightarrow$ b follows from Corollary 4.3.3.21 c $\Rightarrow$ a. ■

Example 4.3.3.23 *Let* $n \in \mathbb{N}$ *and* $E \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. *Then* $\operatorname{Un} E_{n,n}$ *is the set of extreme elements of* $(E_{n,n})^{\#}$.

Let u be an extreme element of $(E_{n,n})^{\#}$. Define

$$u := x^*x\,, \quad v := xx^*\,,$$

$$[\alpha_{ij}]_{i,j\in\mathbb{N}_n} := 1 - u\,, \quad [\beta_{ij}]_{i,j\in\mathbb{N}_n} := 1 - v\,.$$

Take $p, q \in \mathbb{N}_n$ and put

$$y := [\delta_{ip}\delta_{jq}]_{i,j\in\mathbb{N}_n}\,.$$

Then

$$(1-v)y = \left[\sum_{k=1}^{n} \beta_{ik}\delta_{kp}\delta_{jq}\right]_{i,j\in\mathbb{N}_n} = [\beta_{ip}\delta_{jq}]_{i,j\in\mathbb{N}_n}\,.$$

By Theorem 4.3.3.20 a $\Rightarrow$ b,

$$0 = (1-v)y(1-u) = \left[\sum_{k=1}^{n} \beta_{ip}\delta_{kq}\alpha_{kj}\right]_{i,j\in\mathbb{N}_n} = [\beta_{ip}\alpha_{qj}]_{i,j\in\mathbb{N}_n}\,.$$

Hence

$$\beta_{ip}\alpha_{qj} = 0$$

for all $i, j, p, q \in \mathbb{N}_n$. Assume $\beta_{ip} \neq 0$ for some $i, p \in \mathbb{N}_n$. Then $\alpha_{qj} = 0$ for all $q, j \in \mathbb{N}_n$, i.e. $u = 1$. By Proposition 2.1.2.13, $v = 1$, so that $x \in \operatorname{Un} E_{n,n}$. If $\beta_{ip} = 0$ for all $i, p \in \mathbb{N}_n$, then $v = 1$ and, as above, $u = 1$ and $x \in \operatorname{Un} E_{n,n}$.

The converse implication follows from Theorem 4.3.3.20 b $\Rightarrow$ a. ■

Remark. The left and the right shift of ℓ^2 are not unitary elements of $\mathcal{L}(\ell^2)$, but they are extreme points of $\mathcal{L}(\ell^2)^{\#}$ (Example 5.3.1.20). This shows that not all C^*-algebras have the above property of $E_{n,n}$. It will be shown (Remark of Corollary 6.3.6.5), that for every finite-dimensional C^*-algebra E, $\operatorname{Un} E$ is the set of extreme points of $E^{\#}$.

4.3.4 Hereditary C^*–Subalgebras

Definition 4.3.4.1 (0) *A subset F of E is called* ***hereditary*** *if for every $x \in E_+$, if $x \leq y$ for some $y \in F$, then $x \in F$.*

$\{0\}$ and E are hereditary C^*–subalgebras of E. The intersection of any nonempty family of hereditary C^*–subalgebras of E is hereditary. If A is a subset of E, then the smallest hereditary C^*–subalgebra of E containing A is called the ***hereditary C^*–subalgebra generated by A.***

If E is order complete (order σ–complete) then every hereditary C^*–subalgebra of E is order complete (order σ–complete).

Example 4.3.4.2 (0) *Given $p \in \Pr E$, pEp is a hereditary C^*–subalgebra of E.*

Take $x \in E_+$ and $y \in E$ with

$$x \leq pyp\,.$$

Then we get successively that

$$0 \leq (1-p)x(1-p) \leq (1-p)pyp(1-p) = 0$$

(Corollary 4.2.2.3),

$$\left\| x^{\frac{1}{2}}(1-p) \right\|^2 = \|(1-p)x(1-p)\| = 0\,,$$

$$x^{\frac{1}{2}}(1-p) = 0\,,$$

$$x(1-p) = 0\,,$$

$$x = xp = px\,,$$

$$x = pxp \in pEp\,. \qquad \blacksquare$$

Proposition 4.3.4.3 *A convex subcone of E_+ is a face of E_+ iff it is hereditary.*

Let F be a convex subcone of E_+.

First suppose that F is a face and take $(x,y) \in E_+ \times F$ with $x \leq y$. Then

$$x, y - x \in E_+, \quad x + (y - x) = y \in F.$$

Since F is a face of E_+, $x \in F$. Hence F is hereditary.

Now suppose that F is hereditary and take $x, y \in E_+$ with $x + y \in F$. It follows from

$$x \leq x + y, \quad y \leq x + y$$

that $x, y \in F$. Hence F is a face of E_+. ■

Corollary 4.3.4.4 *If F is a hereditary C^*-subalgebra of E_+, then a convex subcone of F_+ is a face of F_+ iff it is a face of E_+. In particular, given $p \in \Pr E$, pE_+p is a face of E_+.* ■

Proposition 4.3.4.5 (0) *Let $\mathfrak{F}$ be the set of closed left ideals and $\mathfrak{G}$ the set of hereditary C^*-subalgebras of E.*

a) *Given $F \in \mathfrak{F}$, $F \cap F^* \in \mathfrak{G}$. Define*

$$\varphi : \mathfrak{F} \longrightarrow \mathfrak{G}, \quad F \longmapsto F \cap F^*.$$

b) *Given $G \in \mathfrak{G}$, $\{x \in E \mid x^*x \in G\} \in \mathfrak{F}$. Define*

$$\psi : \mathfrak{G} \longrightarrow \mathfrak{F}, \quad G \longmapsto \{x \in E \mid x^*x \in G\}.$$

c) *Given $F_1, F_2 \in \mathfrak{F}$,*

$$F_1 \subset F_2 \Longleftrightarrow \varphi(F_1) \subset \varphi(F_2) \Longleftrightarrow \varphi(F_1)_+ \subset F_2.$$

d) *Given $F \in \mathfrak{F}$, $\psi(\varphi(F)) = F$. Given $G \in \mathfrak{G}$,*

$$\operatorname{Re}\varphi(\psi(G)) \subset G \subset \varphi(\psi(G)).$$

e) *Every closed ideal of E is a hereditary C^*-subalgebra of E.*

a) It is obvious that $F \cap F^*$ is a C^*-subalgebra of E. Take

$$(x,y) \in E_+ \times (F \cap F^*)$$

with $x \leq y$. By Proposition 4.2.6.1, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $F_+^{\#}$ such that

$$\lim_{n\to\infty} yx_n = y\,.$$

It follows successively that

$$0 \leq (1-x_n)x(1-x_n) \leq (1-x_n)y(1-x_n)$$

(Corollary 4.2.2.3),

$$\left\|x^{\frac{1}{2}} - x^{\frac{1}{2}}x_n\right\|^2 = \left\|x^{\frac{1}{2}}(1-x_n)\right\|^2 = \|(1-x_n)x(1-x_n)\| \leq$$

$$\leq \|(1-x_n)y(1-x_n)\| \leq \|y(1-x_n)\| = \|y - yx_n\|$$

(Corollaries 4.2.1.17 a $\Rightarrow$ c, 4.2.1.18),

$$x^{\frac{1}{2}} = \lim_{n\to\infty} x^{\frac{1}{2}}x_n \in F\,,$$

$$x = x^{\frac{1}{2}}x^{\frac{1}{2}} \in F\,,$$

$$x \in F \cap F^*\,,$$

$$F \cap F^* \in \mathfrak{G}\,.$$

b) Take $x, y \in \psi(G)$. Then

$$(x+y)^*(x+y) = x^*x + y^*y + x^*y + y^*x \leq 2(x^*x + y^*y) \in G$$

(Corollary 4.2.2.4). Hence $x + y \in \psi(G)$ and $\psi(G)$ is thus a vector subspace of E.

Take $(x, y) \in E \times \psi(G)$. Then

$$(xy)^*(xy) = y^*x^*xy \leq \|x\|^2 y^*y$$

(Corollaries 4.2.1.18, 4.2.2.3). Thus $xy \in \psi(G)$ and so $\psi(G)$ is a left ideal of E. Since the map

$$E \longrightarrow E\,, \quad x \longmapsto x^*x$$

is continuous, $\psi(G)$ is closed. Thus $\psi(G) \in \mathfrak{F}$.

c) Assume that $\varphi(F_1)_+ \subset F_2$ and take $x \in F_1$. By Proposition 4.2.6.1, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $F_1 \cap E_+^{\#}$ such that

$$\lim_{n\to\infty} xx_n = x\,.$$

Now, given $n \in \mathbb{N}$,

$$x_n \in F_1 \cap F_1^* = \varphi(F_1)\,,$$

$$x_n \in \varphi(F_1)_+ \subset F_2\,.$$

Thus

$$xx_n \in F_2$$

for every $n \in \mathbb{N}$ and

$$x = \lim_{n\to\infty} xx_n \in F_2\,,$$

$$F_1 \subset F_2\,.$$

The other implications are trivial.

d) Take $x \in G$. Then

$$xx^*\,,\ x^*x \in G\,,$$

and so

$$x, x^* \in \psi(G)\,,$$

$$x \in \varphi(\psi(G))\,,$$

$$G \subset \varphi(\psi(G))\,.$$

Take $y \in \varphi(\psi(G))_+$. Then $y^2 \in G$ and so $y \in G$. Since every element of $\mathrm{Re}\,\varphi(\psi(G))$ is the linear combination of two elements of $\varphi(\psi(G))_+$ (Theorem 4.2.2.9 a)), it follows that

$$\mathrm{Re}\,\varphi(\psi(G)) \subset G\,.$$

By the above considerations,

$$\mathrm{Re}\,\varphi(\psi(\varphi(F))) \subset \varphi(F) \subset \varphi(\psi(\varphi(F)))\,.$$

Thus, by c),

$$\psi(\varphi(F)) \subset F \subset \psi(\varphi(F))\,,$$

$$\psi(\varphi(F)) = F\,.$$

e) By Corollary 4.2.6.2, every closed ideal of E is involutive. The assertion thus follows from a). ■

Proposition 4.3.4.6 (0) *If F is a C^*-subalgebra of E, then the following are equivalent:*

a) *F is hereditary.*

b) *$x \in E_+$, $y \in F_+ \Rightarrow yxy \in F$,*

If $\mathbb{K} = \mathbb{C}$, then the above assertions are equivalent to:

c) *Given $x \in E$ and $y, z \in F$,*

$$yxz \in F\,.$$

a $\Rightarrow$ b & c. By Proposition 4.3.4.5 b),d), there is a left ideal G of E such that

$$\mathrm{Re}\,(G \cap G^*) \subset F \subset G \cap G^*\,.$$

We have that $y^*, z \in G$ and so

$$yxz \in G\,,$$

$$(yxz)^* = z^*x^*y^* \in G\,.$$

If $x \in E_+$, $y \in E_+$ then

$$yxy \in \mathrm{Re}\,(G \cap G^*) \subset F\,.$$

If $\mathbb{K} = \mathbb{C}$, then

$$yxz \in G \cap G^* = F\,.$$

b $\Rightarrow$ a. Take $x \in E_+$ and $y \in F$ with $x \leq y$. Take $z \in F_+$. Then

$$0 \leq (1-z)x(1-z) \leq (1-z)y(1-z)$$

(Corollary 4.2.2.3), so that, by Corollary 4.2.1.18,

$$\left\|x^{\frac{1}{2}}(1-z)\right\|^2 = \|(1-z)x(1-z)\| \leq \|(1-z)y(1-z)\| = \left\|y^{\frac{1}{2}}(1-z)\right\|^2\,.$$

Let $\mathfrak{F}$ be the canonical approximate unit of F (Theorem 4.2.8.2). Then

$$0 \leq \limsup_{z,\mathfrak{F}} \left\|x^{\frac{1}{2}}(1-z)\right\|^2 \leq \lim_{z,\mathfrak{F}} \left\|y^{\frac{1}{2}}(1-z)\right\|^2 = 0\,.$$

Thus

$$x^{\frac{1}{2}} = \lim_{z,\mathfrak{F}} x^{\frac{1}{2}} z = \lim_{z,\mathfrak{F}} z x^{\frac{1}{2}} ,$$

$$x = x^{\frac{1}{2}} x^{\frac{1}{2}} = \lim_{z,\mathfrak{F}} z x^{\frac{1}{2}} x^{\frac{1}{2}} z = \lim_{z,\mathfrak{F}} z x z \in F .$$

Hence F is hereditary.

c $\Rightarrow$ b is trivial. ■

Corollary 4.3.4.7 *Assume* $\mathbb{K} = \mathbb{C}$. *If* $x \in E_+$, *then* $\overline{xEx}$ *is the hereditary* C^**-subalgebra of* E *generated by* x.

It is easy to see that $\overline{xEx}$ is a C^*-subalgebra of E. By Proposition 4.3.4.6 b $\Rightarrow$ a, it is hereditary. Let $\mathfrak{F}$ be an approximate unit of E (Theorem 4.2.8.2). Then

$$x^2 = \lim_{y,\mathfrak{F}} xyx \in \overline{xEx} ,$$

so that

$$x = (x^2)^{\frac{1}{2}} \in \overline{xEx} .$$

Let F be a hereditary C^*-subalgebra of E containing x. By Proposition 4.3.4.6 a $\Rightarrow$ c,

$$xEx \subset F ,$$

and so

$$\overline{xEx} \subset F .$$

Hence $\overline{xEx}$ is the hereditary C^*-subalgebra of E generated by x. ■

Proposition 4.3.4.8 *Let* F *be a hereditary* C^**-subalgebra of* E *and take* $x \in E_+$. *If* E *is unital and if for each* $\varepsilon > 0$ *there is a* $y \in F_+$ *such that*

$$x \leq y + \varepsilon 1 ,$$

then $x \in F$.

Take $\varepsilon > 0$. By hypothesis, there is a $y_\varepsilon \in F_+$ such that

$$x \leq y_\varepsilon^2 + \varepsilon^2 1 \leq y_\varepsilon^2 + 2\varepsilon y_\varepsilon + \varepsilon^2 1 = (y_\varepsilon + \varepsilon 1)^2 .$$

Hence

$$0 \leq (y_\varepsilon + \varepsilon 1)^{-1} x (y_\varepsilon + \varepsilon 1)^{-1} \leq 1$$

(Corollary 4.2.2.3) and

$$\|(y_\varepsilon + \varepsilon 1)^{-1} x (y_\varepsilon + \varepsilon 1)^{-1}\| \leq 1$$

(Corollary 4.2.1.17 b $\Rightarrow$ a). It follows from

$$(y_\varepsilon + \varepsilon 1) - y_\varepsilon = \varepsilon 1$$

that

$$1 - y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1} = \varepsilon (y_\varepsilon + \varepsilon 1)^{-1} ,$$

so that

$$\left\| x^{\frac{1}{2}} - x^{\frac{1}{2}} y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1} \right\|^2 = \left\| x^{\frac{1}{2}} (1 - y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1}) \right\|^2 =$$

$$= \left\| x^{\frac{1}{2}} \varepsilon (y_\varepsilon + \varepsilon 1)^{-1} \right\|^2 = \varepsilon^2 \|(y_\varepsilon + \varepsilon 1)^{-1} x (y_\varepsilon + \varepsilon 1)^{-1}\| \leq \varepsilon^2 .$$

Hence

$$x^{\frac{1}{2}} = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} x^{\frac{1}{2}} y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1} = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} (y_\varepsilon + \varepsilon 1)^{-1} y_\varepsilon x^{\frac{1}{2}} ,$$

$$x = x^{\frac{1}{2}} x^{\frac{1}{2}} = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} (y_\varepsilon + \varepsilon 1)^{-1} y_\varepsilon x y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1} .$$

Since

$$(y_\varepsilon + \varepsilon 1)^{-1} y_\varepsilon = y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1} \in F_+$$

(Corollary 4.2.1.12), it follows by Proposition 4.3.4.6 a $\Rightarrow$ b, that

$$(y_\varepsilon + \varepsilon 1)^{-1} y_\varepsilon x y_\varepsilon (y_\varepsilon + \varepsilon 1)^{-1} \in F$$

for every $\varepsilon > 0$. Hence $x \in F$. ■

Proposition 4.3.4.9 *Assume E unital. Let F be a unital C^*-subalgebra of E and G a hereditary C^*-subalgebra of E such that $G \subset F$ and such that every positive linear form on E which vanishes on G vanishes on all of F. Then* $\operatorname{Re} F \subset G$.

Take $x \in F_+$ and $\varepsilon > 0$. Put

$$K' := \{x' \in \tau(F) \mid x'(x) \geq \varepsilon\}.$$

K' is a compact set of $\tau(F)$ (Proposition 2.3.5.9 a)). For each $x' \in K'$, there is a $y_{x'} \in G_+$ such that

$$x'(y_{x'}) > 0.$$

Given $x' \in K'$, define

$$U_{x'} := \{y' \in K' \mid y'(y_{x'}) > 0\}.$$

Then $(U_{x'})_{x' \in K'}$ is an open covering of K'. Since K' is compact, there is a finite subset A' of K' such that

$$K' \subset \bigcup_{x' \in A'} U_{x'}.$$

We put

$$y := \sum_{x' \in A'} y_{x'}.$$

Since K' is compact,

$$\alpha := \inf_{x' \in K'} x'(y) > 0.$$

Let

$$z := \frac{\|x\|}{\alpha} y.$$

Take $x' \in \tau(F)$. If $x'(x) \geq \varepsilon$, then $x' \in K'$ so that

$$x'(z + \varepsilon 1) \geq x'(z) = \frac{\|x\|}{\alpha} x'(y) \geq \|x\| \geq x'(x).$$

If $x'(x) < \varepsilon$, then

$$x'(x) < \varepsilon \leq x'(z + \varepsilon 1).$$

Hence

$$x \leq z + \varepsilon 1$$

(Theorem 4.2.2.1 d $\Rightarrow$ a). By Proposition 4.3.4.8, $x \in G$. Hence $F_+ \subset G$ and therefore $\operatorname{Re} F \subset G$. (Theorem 4.2.2.9 a)). ■

Proposition 4.3.4.10 *Assume* $\mathbb{K} = \mathbb{C}$. *Let* F *be a hereditary* C^**-subalgebra of* E *and* G *a closed ideal of* F. *Then there is a closed ideal* H *of* E *such that*

$$G = F \cap H .$$

Given subsets A, B of E, define

$$AB := \{xy \mid (x, y) \in A \times B\},$$

$$\widetilde{A} := \text{the vector subspace of } E \text{ generated by } A .$$

Further, let

$$H := \widetilde{\overline{EGE}}.$$

Step 1 H is a closed ideal of E containing G

It is obvious that $\widetilde{\overline{EGE}}$ is an ideal of E, so H is a closed ideal of E. Using the approximate unit of E (Theorem 4.2.8.2), we see that H contains G.

Step 2 $F \cap H \subset \overline{FHF}$, $G \subset \overline{G^3}$

This follows from the existence of approximate units (Theorem 4.2.8.2).

Step 3 $FEF \subset F$

The assertion follows from Proposition 4.3.4.6 a $\Rightarrow$ c.

Step 4 $FEG^3EF \subset G$

By Step 3,

$$FEG^3EF = (FEG)G(GEF) \subset FGF \subset G .$$

Step 5 $G = F \cap H$

By Steps 2 and 4,

$$FEGEF \subset \overline{FEG^3EF} \subset G ,$$

so, by Step 2

$$F \cap H \subset \overline{FHF} \subset \overbrace{\overline{FEGEF}} \subset G .$$

By Step 1,

$$G \subset F \cap H ,$$

so that

$$G = F \cap H . \qquad \blacksquare$$

Proposition 4.3.4.11 (0) *Let E be $\mathbb{C}$-order σ-complete. Take $x \in \operatorname{Sn} E$ and let F be the hereditary C^*-subalgebra of E generated by x. If f is a bounded Borel function on $\sigma(x)$ such that*

$$0 \in \sigma(x) \Longrightarrow \lim_{\substack{\alpha \to 0 \\ \alpha \in \sigma(x)}} f(\alpha) = 0 ,$$

then $f(x) \in F$.

We may assume that $0 \in \sigma(x)$, since otherwise $1 \in E(x) \subset F$ and this implies that

$$F = E .$$

First take f to be real and positive. Then there is a $g \in C(\sigma(x))_+$ such that

$$f \leq g , \quad g(0) = 0 .$$

By Theorem 4.1.3.1 d), $g(x) \in E(x) \subset F$ and by Corollary 4.3.2.5 a),b),

$$0 \leq f(x) \leq g(x) .$$

Since F is hereditary, $f(x) \in F$.

If f is real, then

$$f(x) = f^+(x) - f^-(x) \in F ,$$

by the above result. If $\mathbb{K} = \mathbb{C}$, then

$$f(x) = (\operatorname{re} f)(x) + i(\operatorname{im} f)(x) \in F$$

for an arbitrary f. $\blacksquare$

Remark. We cannot replace the condition

$$0 \in \sigma(x) \Longrightarrow \lim_{\substack{\alpha \to 0 \\ \alpha \in \sigma(x)}} f(\alpha) = 0$$

by the weaker one $f(0) = 0$ as shown by taking $E = \ell^\infty$,

$$x : \mathbb{N} \longrightarrow \mathbb{K}, \quad n \longmapsto \frac{1}{n},$$

and

$$f := e^{\sigma(x)}_{\sigma(x)\setminus\{0\}} .$$

Proposition 4.3.4.12 *Assume E unital. Let F be a hereditary C^*-subalgebra of E and $y' \in F'_+$. Then there is a unique $x' \in E'_+$ such that*

$$x'|F = y', \quad \|x'\| = \|y'\| .$$

If $\mathfrak{F}$ is an approximate unit of F, then

$$x'(x) = \lim_{y,\mathfrak{F}} y'(y^*yxy^*y)$$

for every $x \in \operatorname{Re} E$ (Proposition 4.3.4.6 a $\Rightarrow$ b).

The existence follows from Corollary 4.2.8.9.

We now prove the last assertion (which implies the uniqueness). Now

$$\lim_{y,\mathfrak{F}} x'(1 - y^*y) = x'(1) - \lim_{y,\mathfrak{F}} x'(y^*y) = \|x'\| - \lim_{y,\mathfrak{F}} y'(y^*y) = \|x'\| - \|y'\| = 0$$

(Proposition 2.3.4.10 a)). By Corollary 4.2.1.17 and Proposition 4.2.4.2 e), for every $y \in F^\#$,

$$0 \leq (1 - y^*y)^2 \leq 1 - y^*y$$

so that

$$0 \leq x'((1 - y^*y)^2) \leq x'(1 - y^*y) .$$

It follows that

$$\lim_{y,\mathfrak{F}} x'((1 - y^*y)^2) = 0 .$$

Now

$$|x'(x) - y'(y^*yxy^*y)| \leq |x'(x - y^*yx)| + |x'(y^*yx - y^*yxy^*y)| =$$

$$= |x'((1-y^*y)x)| + |x'(y^*yx(1-y^*y))| \leq$$

$$\leq x'((1-y^*y)^2)^{\frac{1}{2}} x'(x^*x)^{\frac{1}{2}} + x'(y^*yxx^*y^*y)^{\frac{1}{2}} x'((1-y^*y)^2)$$

(Proposition 2.3.4.6 c)). By the above considerations,

$$\lim_{y,\mathfrak{F}} |x'(x) - y'(y^*yxy^*y)| = 0\,,$$

$$x'(x) = \lim_{y,\mathfrak{F}} y'(y^*yxy^*y)\,. \qquad \blacksquare$$

Corollary 4.3.4.13 *Take $p \in \Pr E$ such that pEp is one-dimensional (this happens if $\mathbb{K} = \mathbb{C}$, E is order σ-complete, and p is a minimal element of $\Pr E \backslash \{0\}$).*

a) *There is exactly one $x' \in \tau(E)$ such that*

$$x'(p) = 1\,,$$

and this x' belongs to $\tau_0(E)$.

b) *$px'p = x'$.*

c) *x' is order continuous.*

By Corollary 4.3.2.12 a $\Rightarrow$ d, pEp is one-dimensional if $\mathbb{K} = \mathbb{C}$, E is $\mathbb{C}$-order σ-complete, and p is a minimal element of $\Pr E \backslash \{0\}$.

a) Let y' be the unique element of $\tau(pEp)$. By Example 4.3.4.2, pEp is a hereditary C^*-subalgebra of E, so by Proposition 4.3.4.12, there is a unique $x' \in \tau(E)$ extending y' (which is equivalent to $x'(p) = 1$) and this x' belongs to $\tau_0(E)$.

b) The map

$$E \longrightarrow \mathbb{K}\,, \quad x \longmapsto x'(pxp)$$

belongs to $\tau(E)$. By a), it is x'.

c) Let A be a downward directed set of E with infimum 0. Then pAp is a downward directed set of pEp with infimum 0. Hence by b),

$$\inf_{x \in A} x'(x) = \inf_{x \in A} x'(pxp) = 0$$

and x' is order continuous. $\blacksquare$

Proposition 4.3.4.14 (0) *Let F be an E-module. Then for every $a \in F$, $\{x \in E | xa = x^*a = 0\}$ (resp. $\{x \in E \mid ax = ax^* = 0\}$) is a hereditary C^*-subalgebra of E.*

Put

$$G := \{x \in E | xa = x^*a = 0\} \text{ (resp. } G := \{x \in E | ax = ax^* = 0\}\text{).}$$

It is easy to see that G is a C^*-subalgebra of E. Take $x \in E$ and $y, z \in G$. Then

$$(yxz)a = (yx)(za) = 0 \qquad (a(yxz) = (ay)(xz) = 0)\,,$$

$$(yxz)^*a = z^*x^*y^*a = 0 \qquad (a(yxz)^* = az^*x^*y^* = 0)\,,$$

i.e. $yxz \in G$. By Proposition 4.3.4.6 b $\Rightarrow$ a, G is a hereditary C^*-subalgebra of E. ■

4.3.5 Simple C^*–algebras

Definition 4.3.5.1 (0) *E is called **simple** if $\{0\}$ and E are its only closed ideals.*

Proposition 4.3.5.2 *Every hereditary C^*–subalgebra of a simple complex C^*–algebra is simple.*

Let E be a simple complex C^*–algebra and F a hereditary C^*–subalgebra of E. Further let G be a closed ideal of F. By Proposition 4.3.4.10, there is a closed ideal H of E such that

$$G = F \cap H .$$

Since E is simple, H is $\{0\}$ or E. Hence G is $\{0\}$ or F and thus F is simple as well. ■

Proposition 4.3.5.3 (0) *The following are equivalent for every real C^*–algebra E:*

a) $\overset{\circ}{E}$ *is simple.*

b) E *is a simple purely real C^*–algebra.*

a $\Rightarrow$ b. Let F be a closed non–zero ideal of E. Then $F \times F$ is a closed ideal of $\overset{\circ}{E}$, so that $F \times F = \overset{\circ}{E}$. Let $z \in E$. There is an $(x, y) \in F \times F$ such that $(z, 0) = (x, y)$. We get $z = x \in F$, i.e. $E = F$. Hence E is simple.

By Theorem 4.1.1.8 b), E is a purely real C^*–algebra.

b $\Rightarrow$ a. Assume $\overset{\circ}{E}$ is not simple. Then there is a closed ideal F of $\overset{\circ}{E}$, $\{0\} \neq F \neq \overset{\circ}{E}$. Put

$$G := \{x \in E \mid (0, x) \in F\} .$$

Then G is a closed ideal of E. Assume that $G \neq \{0\}$. Then $G = E$ and $\{0\} \times E \subset F$. It follows that

$$E \times \{0\} = i(\{0\} \times E) \subset F ,$$

$$\overset{\circ}{E} = (E \times \{0\}) + (\{0\} \times E) \subset F ,$$

which is a contradiction. Hence $G = \{0\}$.

Put

$$H := \{x \in E \mid \exists y \in E\,,\ (x,y) \in F\}\,.$$

Take $x \in H$ and $y, z \in E$ such that $(x,y),(x,z) \in F$. Then

$$(0,(y-z)) = (x,y) - (x,z) \in F\,,$$

$$y - z \in G\,,$$

$$y = z\,,$$

by the above considerations. Hence, for every $x \in H$, there is a unique $x' \in E$ such that $(x,x') \in F$.

Let $x, y \in H$, $z \in E$, and $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha x + \beta y, \alpha x' + \beta y') = \alpha(x,x') + \beta(y,y') \in F\,,$$

$$(xz, x'z) = (x,x')(z,0) \in F\,,$$

$$(zx, zx') = (z,0)(x,x') \in F\,,$$

$$(x', -x) = -i(x,x') \in F\,,$$

$$(x^*, -x'^*) = (x,x')^* \in F\,.$$

Hence H is an involutive ideal of E, the map

$$H \longrightarrow H\,, \quad x \longmapsto x'$$

is linear, and

$$x'' = -x\,, \quad x^{*\prime} = -x'^*\,,$$

$$(xy)' = x'y\,, \quad (yx)' = yx'$$

for all $(x,y) \in H \times E$.

Take $x \in H$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\|\alpha x + \beta x'\|^2 = \|(\alpha x^* + \beta x'^*)(\alpha x + \beta y')\| = \|(\alpha x^* - \beta x^{*\prime})(\alpha x + \beta x')\| =$$

$$= \|\alpha^2 x^* x + \alpha\beta(x^* x' - x^{*\prime} x) - \beta^2 x^{*\prime} x'\| =$$

$$= \|\alpha^2 x^* x + \alpha\beta(x^* x' - (x^* x)') - \beta^2 (x^* x')\| =$$

$$= \|\alpha^2 x^* x + \alpha\beta(x^* x' - x^* x') - \beta^2 x^* x''\| =$$

$$= \|\alpha^2 x^* x + \beta^2 x^* x\| = (\alpha^2 + \beta^2)\|x\|^2 .$$

By taking $\alpha = 0$ and $\beta = 1$, we get $\|x'\| = \|x\|$. Hence H is closed and so $H = E$.

Given $\alpha, \beta \in \mathbb{R}$ and $x \in E$, put

$$(\alpha + i\beta)x := \alpha x + \beta x' .$$

By the above considerations

$$\|(\alpha + i\beta)x\| = |\alpha + i\beta|\,\|x\| .$$

For all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $x, y \in E$,

$$((\alpha + i\beta) + (\gamma + i\delta))x = ((\alpha + \gamma) + i(\beta + \delta))x = (\alpha + \gamma)x + (\beta + \delta)x' =$$

$$= (\alpha x + \beta x') + (\gamma x + \delta x') = (\alpha + i\beta)x + (\gamma + i\delta)x ,$$

$$(\alpha + i\beta)(x + y) = (\alpha + i\beta)x + (\alpha + i\beta)y ,$$

$$(\alpha + i\beta)((\gamma + i\delta)x) = (\alpha + i\beta)(\gamma x + \delta x') = \alpha\gamma x + \alpha\delta x' + \beta\gamma x' + \beta\delta x'' =$$

$$= (\alpha\gamma - \beta\delta)x + (\alpha\delta + \beta\gamma)x' = ((\alpha + i\beta)(\gamma + i\delta))x ,$$

$$(\alpha + i\beta)(xy) = \alpha xy + \beta(xy)' = \alpha xy + \beta x' y = ((\alpha + i\beta)x)y ,$$

$$(\alpha + i\beta)(xy) = \alpha xy + \beta xy' = x((\alpha + i\beta)y) ,$$

$$((\alpha + i\beta)x)^* = (\alpha x + \beta x')^* = \alpha x^* - \beta x^{*\prime} = (\alpha - i\beta)x^* .$$

Hence E endowed with the map

$$\mathbb{C} \times E \longrightarrow E , \quad (\alpha + i\beta, x) \longmapsto (\alpha + i\beta)x$$

is an involutive complex Banach algebra and so a complex C^*-algebra. This contradicts b), so $\overset{\circ}{E}$ is simple. ∎

Remark. The last part of the proof can also be deduced from Proposition 2.3.1.43 d).

Proposition 4.3.5.4 (0) *Assume that* $\mathbb{K} = \mathbb{C}$ *and let* F *denote the underlying real* C^**-algebra of* E *(Theorem 4.1.1.8 a)). Then any closed ideal of* F *is a closed ideal of* E. *Hence if* E *is simple,* F *is simple too.*

Let G be a closed ideal of F and take $x \in G$. Further, let $\mathfrak{F}$ be an approximate unit of E. Then

$$ix = \lim_{y,\mathfrak{F}} i(yx) = \lim_{y,\mathfrak{F}} (iy)x \in G .$$

Thus G is a vector subspace of E and therefore a closed ideal of E. ■

Proposition 4.3.5.5 (0) *Let* $(E_\iota)_{\iota \in I}$ *and* $(F_\lambda)_{\lambda \in L}$ *be families of non-degenerate simple* C^**-algebras, and* E *and* F *the* C^**-direct products(sums) of these families, respectively. Given* $\iota \in I$ *and* $\lambda \in L$, *define*

$$\widetilde{E}_\iota := \{x \in E \mid \iota' \in I \backslash \{\iota\} \Longrightarrow x_{\iota'} = 0\}$$

$$\widetilde{F}_\lambda := \{y \in F \mid \lambda' \in L \backslash \{\lambda\} \Longrightarrow y_{\lambda'} = 0\}$$

for every $\iota \in I$ *and* $\lambda \in L$. *If* $u : E \to F$ *is an isomorphism of* C^**-algebras, then there is a bijective map* $\varphi : I \to L$ *such that*

$$\widetilde{F}_{\varphi(\iota)} = u(\widetilde{E}_\iota)$$

for every $\iota \in I$.

Take $\iota \in I$. There is a $\lambda \in L$ such that $u(\widetilde{E}_\iota) \cap \widetilde{F}_\lambda \neq \{0\}$. Since $\widetilde{E}_\iota$ is a closed ideal of E, $u(\widetilde{E}_\iota)$ a closed ideal of F and $u(\widetilde{E}_\iota) \cap \widetilde{F}_\lambda$ a closed ideal of $\widetilde{F}_\lambda$. Since $\widetilde{F}_\lambda$ is simple, $\widetilde{F}_\lambda \subset u(\widetilde{E}_\iota)$. Repeating the above argument for u^{-1}, we get $\widetilde{E}_\iota \subset u^{-1}(\widetilde{F}_\lambda)$, so that $u(\widetilde{E}_\iota) = \widetilde{F}_\lambda$. We put $\varphi(\iota) := \lambda$. The map $\varphi : I \to L$ is obviously injective. By symmetry, it is bijective. ■

Proposition 4.3.5.6 (0) *Let* $(E_\iota)_{\iota \in I}$ *be a family of non-degenerate simple complex* C^**-algebras,* E *the* C^**-direct product (sum) of it,* $\mathcal{F}$ *the set of real* C^**-subalgebras* F *of* E *such that the map*

$$\overset{\circ}{F} \longrightarrow E , \quad (x, y) \longmapsto x + iy$$

is an isomorphism of complex C^**-algebras, and* Ω *the set of pairs* $(\varphi, (u_\iota)_{\iota \in I})$ *such that* φ *is an involution of* I *and for every* $\iota \in I$

$$u_\iota : E_\iota \longrightarrow E_{\varphi(\iota)}$$

is an isomorphism of real C^*-algebras, which is conjugate linear and such that $u_\iota^{-1} = u_{\varphi(\iota)}$. Put

$$E_\omega := \{x \in E \mid \iota \in I \Longrightarrow x_{\varphi(\iota)} = u_\iota x_\iota\}$$

for every $\omega = (\varphi, (u_\iota)_{\iota \in I}) \in \Omega$. Then $E_\omega \in \mathcal{F}$ for every $\omega \in \Omega$ and the map

$$\Omega \longrightarrow \mathcal{F}, \quad \omega \longmapsto E_\omega$$

is bijective.

Let $\mathcal{U}$ be the set of conjugate involutions on E. If we set

$$E_{(u)} := \{x \in E \mid ux = x\}$$

for every $u \in \mathcal{U}$, then by Proposition 2.3.1.43 a),b), $E_{(u)} \in \mathcal{F}$ for every $u \in \mathcal{U}$ and the map

$$\mathcal{U} \longrightarrow \mathcal{F}, \quad u \longmapsto E_{(u)}$$

is bijective.

Given $\omega = (\varphi, (u_\iota)_{\iota \in I}) \in \Omega$, define

$$u_\omega : E \longrightarrow E, \quad x \longmapsto (u_{\varphi(\iota)} x_{\varphi(\iota)})_{\iota \in I}.$$

Then $u_\omega \in \mathcal{U}$ and

$$E_{(u_\omega)} = \{x \in E \mid u_\omega x = x\} = \{x \in E \mid \iota \in I \Longrightarrow x_{\varphi(\iota)} = u_\iota x_\iota\} = E_\omega.$$

Hence we have only to show that the map

$$\Omega \longrightarrow \mathcal{U}, \quad \omega \longmapsto u_\omega$$

is bijective.

Take $u \in \mathcal{U}$. By Proposition 4.3.5.4, for every $\iota \in I$, the underlying real C^*-algebra of E_ι is simple. For every $\iota \in I$, put

$$\widetilde{E}_\iota := \{x \in E \mid \lambda \in I \backslash \{\iota\} \Longrightarrow x_\lambda = 0\}.$$

By Proposition 4.3.5.5, there is a bijective map $\varphi : I \to I$ such that

$$\widetilde{E}_{\varphi(\iota)} = u(\widetilde{E}_\iota)$$

for every $\iota \in I$. We denote for every $\iota \in I$, by $u_\iota : E_\iota \to E_{\varphi(\iota)}$, the map obtained from u by identifying E_ι and $E_{\varphi(\iota)}$ with $\widetilde{E}_\iota$ and $\widetilde{E}_{\varphi(\iota)}$, respectively. Since u^2 is the identity map, it follows that φ is an involution and $u_\iota^{-1} = u_{\varphi(\iota)}$ for every $\iota \in I$. Hence $\omega := (\varphi, (u_\iota)_{\iota \in I}) \in \Omega$ and $u_\omega = u$. Thus the map

$$\Omega \longrightarrow \mathcal{U}, \quad \omega \longmapsto u_\omega$$

is surjective. It is easy to see that this map is injective, so it is bijective. ■

Corollary 4.3.5.7 (0) *The following are equivalent for every simple complex C^*-algebra E:*

a) $\overset{\circ}{E}$ *is isomorphic to the direct C^*-product $E \times E$.*

b) *E has a conjugate involution.*

a $\Rightarrow$ b follows from Proposition 4.3.5.6.
b $\Rightarrow$ a follows from Example 2.3.2.35 f). ■

Proposition 4.3.5.8 (0) *Let E be a simple non-degenerate complex C^*-algebra, $\mathcal{F}$ the class of real C^*-algebras F, such that $\overset{\circ}{F}$ and E are isomorphic, $\mathcal{G}$ a subset of $\mathcal{F}$ such that every element of $\mathcal{F}$ is isomorphic to exactly one element of $\mathcal{G}$, E_0 the underlying real C^*-algebra of E, and $\mathcal{H} := \mathcal{G} \cup \{E_0\}$. We assume $\mathcal{F} \neq \emptyset$. Let further I be a set and $\mathcal{P}$ the set of maps $p : \mathcal{H} \to \mathfrak{P}(I)$ such that:*

1) $F, G \in \mathcal{H}$, $F \neq G \Rightarrow p(F) \cap p(G) = \emptyset$.

2) *there is a bijective map $p(E_0) \to I \backslash I_p$, where*

$$I_p := \bigcup_{F \in \mathcal{H}} p(F).$$

Given $p \in \mathcal{P}$ and $\iota \in I_p$, let $F_{p,\iota}$ be the element F of $\mathcal{H}$ with $\iota \in p(F)$, F_p the C^*-direct product (sum) of the family $(F_{p,\iota})_{\iota \in I_p}$, $\widetilde{E}$ the C^*-direct product (sum) of the family $(E)_{\iota \in I}$, and $\widetilde{\mathcal{F}}$ the class of real C^*-algebras $\widetilde{F}$ such that $\overset{\circ}{\widetilde{F}}$ and $\widetilde{E}$ are isomorphic.

a) *For every element $\widetilde{F}$ of $\widetilde{\mathcal{F}}$, there is a $p \in \mathcal{P}$ such that $\widetilde{F}$ is isomorphic to F_p.*

b) *$p \in \mathcal{P} \Rightarrow F_p \in \widetilde{\mathcal{F}}$.*

c) *The following are equivalent for all $p, q \in \mathcal{P}$:*

c_1) *F_p and F_q are isomorphic.*

c_2) *$F \in \mathcal{H} \Rightarrow \operatorname{Card} p(F) = \operatorname{Card} q(F)$.*

d) *We define a map $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ inductively by*

$$\varphi(1, n) = \frac{1}{2}\left(n + 1 + \cos^2 \frac{n\pi}{2}\right),$$

$$\varphi(m,n) = 1 + \sum_{k=0}^{n-1} \varphi(m-1, n-k)$$

for all $n \in \mathbb{N}$ *and* $m \in \mathbb{N}\backslash\{1\}$. *Let* $\widetilde{\mathcal{G}}$ *be a subset of* $\widetilde{\mathcal{F}}$ *such that every element of* $\widetilde{\mathcal{F}}$ *is isomorphic to exactly one element of* $\widetilde{\mathcal{F}}$. *If*

$$m := \operatorname{Card} \mathcal{G} \in \mathbb{N}, \quad n := \operatorname{Card} I \in \mathbb{N},$$

then

$$\operatorname{Card} \widetilde{\mathcal{G}} = \varphi(m,n).$$

a) We may assume that $\widetilde{F}$ is a real C^*–subalgebra of $\widetilde{E}$ such that the map

$$\overset{\circ}{\widetilde{F}} \longrightarrow \widetilde{E}, \quad (x,y) \longmapsto x + iy$$

is an isomorphism of complex C^*–algebras (Proposition 2.3.1.43 c)).

By Proposition 4.3.5.6, there is an involution φ of I and a family $(u_\iota)_{\iota \in I}$ of conjugate involution $E \to E$ with $u_\iota^{-1} = u_{\varphi(\iota)}$ for every $\iota \in I$ such that

$$\widetilde{F} = \{x \in \widetilde{E} \mid \iota \in I \Longrightarrow x_{\varphi(\iota)} = u_\iota x_\iota\}.$$

Take $\iota \in I$ such that $\varphi(\iota) = \iota$. By Proposition 2.3.1.43 a),

$$F_\iota := \{x \in E \mid u_\iota x = x\} \in \mathcal{F}.$$

Hence there is a unique $G_\iota \in \mathcal{G}$ and an isomorphism $v_\iota : G_\iota \to F_\iota$. We define the map $p : \mathcal{H} \to \mathfrak{P}(I)$ as follows: for each $G \in \mathcal{G}$, we put

$$p(G) := \{\iota \in I \mid G = G_\iota\}$$

and take as $p(E_0)$ a subset of $I \backslash \bigcup_{G \in \mathcal{G}} p(G)$ such that

$$\varphi(p(E_0)) = I \backslash \bigcup_{G \in \mathcal{H}} p(G).$$

Then $p \in \mathcal{P}$. For $x \in F_p$ and $\iota \in I$, define

$$x'_\iota := \begin{cases} v_\iota x_\iota & \text{if} \quad \iota \in \bigcup_{G \in \mathcal{G}} p(G) \\ x_\iota & \text{if} \quad \iota \in p(E_0) \\ u_{\varphi(\iota)} x_{\varphi(\iota)} & \text{if} \quad \iota \in I \backslash I_p \end{cases}$$

and put

$$u : F_p \longrightarrow \widetilde{E}\,, \quad x \longmapsto (x'_\iota)_{\iota\in I}\,.$$

u is an injective homomorphism of real C^*-algebras.

Take $x \in F_p$. Then

$$\iota \in \bigcup_{G\in\mathcal{G}} p(G) \Longrightarrow u_\iota x'_\iota = u_\iota v_\iota x_\iota = v_\iota x_\iota = x'_\iota\,,$$
$$\iota \in p(E_0) \Longrightarrow x'_{\varphi(\iota)} = u_{\varphi^2(\iota)} x_{\varphi^2(\iota)} = u_\iota x'_\iota\,,$$
$$\iota \in I\backslash I_p \Longrightarrow u_\iota x'_\iota = u_\iota u_{\varphi(\iota)} x_{\varphi(\iota)} = x_{\varphi(\iota)} = x'_{\varphi(\iota)}\,.$$

Hence $ux \in \widetilde{F}$, $u(F_p) \subset \widetilde{F}$.

Take $y \in \widetilde{F}$. Then

$$y_{\varphi(\iota)} = u_\iota y_\iota$$

for every $\iota \in I$. For $\iota \in I_p$, put

$$x_\iota := \begin{cases} v_\iota^{-1} y_\iota & \text{if } \iota \in \bigcup\limits_{G\in\mathcal{G}} p(G) \\ y_\iota & \text{if } \iota \in p(E_0)\,. \end{cases}$$

Then $x := (x_\iota)_{\iota\in I} \in F_p$ and for every $\iota \in I\backslash I_p$,

$$x'_\iota = u_{\varphi(\iota)} x_{\varphi(\iota)} = u_{\varphi(\iota)} y_{\varphi(\iota)} = y_\iota\,.$$

Hence $ux = y$, $\widetilde{F} \subset u(F_p)$, and $\widetilde{F}$ is isomorphic to F_p.

b) Since $\mathcal{F} \neq \emptyset$, there is an isomorphism of complex C^*-algebras

$$u : \overset{\circ}{E} \longrightarrow E \times E$$

(Corollary 4.3.5.7 b $\Rightarrow$ a). Put

$$\begin{cases} \pi_1 : E \times E \longrightarrow E\,, & (x,y) \longmapsto x \\ \pi_2 : E \times E \longrightarrow E\,, & (x,y) \longmapsto y\,. \end{cases}$$

For every $\iota \in p(E_0)$ put $u_\iota := u$ and for every $\iota \in \bigcup\limits_{G\in\mathcal{G}} p(G)$ let

$$u_\iota : \overset{\circ}{F}_{p,\iota} \longrightarrow E$$

be an isomorphism of complex C^*-algebra. Let

$$\varphi : I\backslash I_p \longrightarrow p(E_0)$$

be a bijection. Given $x \in F_p$ and $\iota \in I_p$ define

$$x'_\iota := \begin{cases} u_\iota x_\iota & \text{in } \iota \in \bigcup_{G \in \mathfrak{G}} p(G) \\ \pi_1 u x_\iota & \text{in } \iota \in \varphi(E_0) \\ \pi_2 u x_{\varphi(\iota)} & \text{in } \iota \in I \backslash I_p . \end{cases}$$

Then the map

$$\overset{\circ}{F}_p \longrightarrow \widetilde{E}, \quad x \longmapsto (x'_\iota)_{\iota \in I_p}$$

is an isomorphism of complex C^*-algebras, so C^*-algebras, so $F_p \in \mathcal{F}$.

$c_1 \Rightarrow c_2$. By Proposition 4.3.5.3 a $\Rightarrow$ b, all C^*-algebras of $\mathcal{G}$ are simple and none is isomorphic to E_0. By Proposition 4.3.5.4, E_0 is simple. Let $u : E_p \to E_q$ be an isomorphism. By Proposition 4.3.5.5, there is a bijective map $\varphi : I_p \to I_q$ such that $F_{p,\iota}$ and $F_{q,\varphi(\iota)}$ are isomorphic for every $\iota \in I_p$. Let $F \in \mathcal{H}$, $\iota \in p(F)$, and $G \in \mathcal{H}$ such that $\varphi(\iota) \in q(G)$. Then $F_{p,\iota} = F$ and $F_{q,\varphi(\iota)} = G$, so that F and G are isomorphic. It follows that $F = G$, $\varphi(\iota) \in q(F)$, $\varphi(p(F)) \subset q(F)$. By symmetry, $\varphi(p(F)) = q(F)$. Hence

$$\operatorname{Card} p(F) = \operatorname{Card} q(F) .$$

$c_2 \Rightarrow c_1$ is obvious.

d) follows from a),b), and c). ■

Proposition 4.3.5.9 *Let E, F be C^*-algebras, G a C^*-subalgebra of $E \times F$ and E_0 a simple C^*-subalgebra of E such that:*

1) *for every $x \in E_0$, there is a $y \in F$ with $(x, y) \in G$.*

2) *there is an $x_0 \in E_0 \backslash \{0\}$ with $(x_0, 0) \in G$.*

Then $E_0 \times \{0\} \subset G$.

Define

$$H := \{x \in E_0 \mid (x, 0) \in G\} .$$

Take $x \in H$ and $y \in E_0$. By 1), there is a $z \in F$ such that $(y, z) \in G$. We get

$$(xy, 0) = (x, 0)(y, z) \in G ,$$

$$(yx, 0) = (y, z)(x, 0) \in G ,$$

and so $xy, yx \in H$. Hence H is an ideal of E_0, which is obviously closed. By 2), $H \neq \{0\}$. Since E_0 is simple, $H = E_0$, i.e. $E_0 \times \{0\} \subset G$. ■

Corollary 4.3.5.10 *Let $(E_\iota)_{\iota\in I}$ be a family of C^*-algebras, E its C^*-direct product, and F a C^*-subalgebra of E. For each $\iota \in I$, let F_ι be a simple C^*-subalgebra of E_ι,*

$$p_\iota : E \longrightarrow E_\iota, \quad x \longmapsto x_\iota,$$

and

$$q_\iota : \{x \in F \mid x_\iota \in F_\iota\} \longrightarrow \prod_{\lambda\in I\setminus\{\iota\}} E_\lambda, \quad x \longmapsto (x_\lambda)_{\lambda\in I\setminus\{\iota\}}.$$

If, for every $\iota \in I$, the map q_ι is not injective and $F_\iota \subset p_\iota(F)$, then F contains the C^*-direct sum of the family $(F_\iota)_{\iota\in I}$.

By Proposition 4.3.5.9,

$$F_\iota \times \prod_{\lambda\in I\setminus\{\iota\}} \{0\} \subset F$$

for every $\iota \in I$ and the assertion now follows. ■

Corollary 4.3.5.11 *Let $(E_\iota)_{\iota\in I}$ be a finite family of simple C^*-algebras, E its C^*-directed product, and F a closed ideal of E. Given $J \subset I$, define*

$$p_J : E \longrightarrow \prod_{\iota\in J} E_\iota, \quad x \longmapsto (x_\iota)_{\iota\in J}.$$

If $E_\iota = p_{\{\iota\}}(F)$ for every $\iota \in I$, then there is a $J \subset I$ such that $p_J|F$ is injective and

$$p_J(F) = \prod_{\iota\in J} E_\iota.$$

We prove the assertion by induction on $\operatorname{Card} I$. If $p_{I\setminus\{\iota\}}|F$ is not injective for every $\iota \in I$, then by Corollary 4.3.5.10,

$$F = \prod_{\iota\in I} E_\iota.$$

So take $\iota \in I$ with $p_{I\setminus\{\iota\}}|F$ injective. By the induction hypothesis, there is a $J \subset I\setminus\{\iota\}$ such that $p_J|p_{I\setminus\{\iota\}}(F)$ is injective and

$$p_J(p_{I\setminus\{\iota\}}(F)) = \prod_{\iota\in J} E_\iota,$$

then $p_J|F$ is injective and

$$p_J(F) = \prod_{\iota\in J} E_\iota.$$ ■

4.3.6 Supplementary Results Concerning Complexification

Proposition 4.3.6.1 (0) *Let E be a real C^*-algebra and*

$$\varphi : E \longrightarrow \overset{\circ}{E}, \quad x \longmapsto (x, 0).$$

Given $x' \in E'$, define

$$\widetilde{x'} : \overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto x'(x) + ix'(y).$$

a) $x' \in E' \Rightarrow \widetilde{x'} \in (\overset{\circ}{E})'$, $\varphi'\widetilde{x'} = x'$, $\widetilde{x'}^* = \widetilde{x'^*}$.

b) *The following are equivalent for every $x' \in E'$:*

b_1) $x' \in E'_+$.

b_2) $\widetilde{x'} \in (\overset{\circ}{E})'_+$.

c) $x, y \in E$, $\theta \in \mathbb{R} \Rightarrow \|(\cos\theta)x + (\sin\theta)y\| \leq \|x^*x + y^*y\|^{\frac{1}{2}} \leq \|(x, y)\|$.

d) $x' \in E' \Rightarrow \|x'\| = \|\widetilde{x'}\|$.

e) *The following are equivalent for every $x' \in E'$:*

e_1) $x' \in \operatorname{Re} E'$.

e_2) $\widetilde{x'} \in \operatorname{Re} (\overset{\circ}{E})'$.

If these conditions are fulfilled then

$$\widetilde{x'^+} = \widetilde{x'}^+, \quad \widetilde{x'^-} = \widetilde{x'}^-.$$

f) *The identity map $\overset{\circ}{E} \to E \times E$ is an isomorphism of real Banach spaces.*

g) *If $\mathfrak{F}$ is an approximate unit of E then for every $x' \in E'$,*

$$x' \in E'_+ \Longleftrightarrow \lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|.$$

a) The relation $\widetilde{x'} \in (\overset{\circ}{E})'$ is easy to check. For every $x \in E$,

$$\langle x, \varphi'\widetilde{x'} \rangle = \langle \varphi x, \widetilde{x'} \rangle = \langle x, x' \rangle.$$

Thus

$$\varphi'\widetilde{x'} = x'.$$

Take $(x,y) \in \overset{\circ}{E}$. Then

$$\widetilde{x'^*}((x,y)) = \overline{\widetilde{x'}((x,y)^*)} = \overline{\widetilde{x'}((x^*,-y^*))} = \overline{x'(x^*) - ix'(y^*)} =$$

$$= x'(x^*) + ix'(y^*) = x'^*(x) + ix'^*(y) = \widetilde{x'^*}((x,y))\,.$$

Hence

$$\widetilde{x'}^* = \widetilde{x'^*}\,.$$

b) By a) and Proposition 4.1.1.27 c),

$$\widetilde{x'} \in (\overset{\circ}{E})'_+ \iff x' = \varphi'\widetilde{x'} \in E'_+\,.$$

c) Take $x' \in \tau(E)$. By b), $\widetilde{x'} \in (\overset{\circ}{E})'_+$ and by a), $\varphi'\widetilde{x'} \in \tau(E)$. By Proposition 4.1.1.27 d) (and Theorem 4.2.8.2), $\widetilde{x'} \in \tau(\overset{\circ}{E})$. We get

$$\|(x,y)\|^2 = \|(x,y)^*(x,y)\| = \|(x^*x + y^*y\,,\, x^*y - y^*x)\| \geq$$

$$\geq |\widetilde{x'}((x^*x + y^*y, x^*y - y^*x))| = |x'(x^*x + y^*y) + ix'(x^*y - y^*x)| =$$

$$= x'(x^*x + y^*y)\,.$$

By Corollary 4.2.8.5 c) and Corollary 4.2.2.7,

$$\|(x,y)\|^2 \geq \|x^*x + y^*y\| \geq \|(\cos\theta)x + (\sin\theta)y\|^2\,.$$

d) We have $\|\varphi\| = 1$, so $\|\varphi'\| = 1$ and by a),

$$\|x'\| = \|\varphi'\widetilde{x'}\| \leq \|\widetilde{x'}\|\,.$$

Take $(x,y) \in \overset{\circ}{E}$. For some $\theta \in \mathbb{R}$,

$$|\widetilde{x'}((x,y))| = \widetilde{x'}((x,y))e^{-i\theta} = \widetilde{x'}((\cos\theta - i\,\sin\theta)(x,y)) =$$

$$= \widetilde{x'}(((\cos\theta)x + (\sin\theta)y)\,,\,(\cos\theta)y - (\sin\theta)x)) =$$

$$= x'((\cos\theta)x + (\sin\theta)y) + ix'((\cos\theta)y - (\sin\theta)x) =$$

$$= x'((\cos\theta)x + (\sin\theta)y) \leq \|x'\|\,\|(\cos\theta)x + (\sin\theta)y\| \leq \|x'\|\,\|(x,y)\|\,,$$

where the last inequality follows from c). Hence

$$\|\widetilde{x'}\| \leq \|x'\|\,, \quad \|\widetilde{x'}\| = \|x'\|\,.$$

e) By a) and Proposition 4.1.1.27 b),

$$\widetilde{x'} \in \operatorname{Re}(\overset{\circ}{E})' \Longleftrightarrow x' = \varphi'\widetilde{x'} \in \operatorname{Re} E'\,.$$

Assume $x' \in \operatorname{Re} E'$. Then $x'^+, x'^- \in E'_+$ and

$$x' = x'^+ - x'^-\,,\ \|x'\| = \|x'^+\| + \|x'^-\|$$

(Theorem 4.2.8.13). By b) and d),

$$\widetilde{x'^+}\,,\ \widetilde{x'^-} \in (\overset{\circ}{E})'_+$$

$$\|\widetilde{x'}\| = \|x'\| = \|x'^+\| + \|x'^-\| = \|\widetilde{x'^+}\| + \|\widetilde{x'^-}\|\,.$$

Since

$$\widetilde{x'} = \widetilde{x'^+} - \widetilde{x'^-}$$

it follows by Theorem 4.2.8.13,

$$\widetilde{x'}^+ = \widetilde{x'^+}\,, \quad \widetilde{x'}^- = \widetilde{x'^-}\,.$$

f) follows from c) and Proposition 4.1.1.27 a).

g) Suppose

$$\lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|\,.$$

By d),

$$\lim_{x,\mathfrak{F}} \widetilde{x'}((x^*x,0)) = \lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\| = \|\widetilde{x'}\|\,.$$

By Proposition 4.1.2.27 i),

$$\widetilde{x'} \in (\overset{\circ}{E})'_+$$

so that by $b_2 \Rightarrow b_1$,

$$x' \in E'_+\,.$$

The reverse implication follows from Proposition 2.3.4.10 a). ■

Corollary 4.3.6.2 (0) *Let E be a real C^*-algebra. For $(x',y') \in \overset{\circ}{\overbrace{E'}}$, define*

$$\widetilde{(x',y')} : \overset{\circ}{E} \longrightarrow \mathbb{C}\,, \quad (x,y) \longmapsto x'(x) - y'(y) + i(x'(y) + y'(x))\,.$$

a) *For every $(x',y') \in \overset{\circ}{E'}$, $\widetilde{(x',y')} \in (\overset{\circ}{E})'$ and*

$$\sup\{\|x'\|, \|y'\|\} \leq \left\|\widetilde{(x',y')}\right\| \leq \|x'\| + \|y'\|\,.$$

b) *If we endow $\overset{\circ}{\overbrace{E'}}$ with the norm*

$$\overset{\circ}{\overbrace{E'}} \longrightarrow \mathbb{R}_+\,, \quad (x',y') \longmapsto \|\widetilde{(x',y')}\|$$

then the map

$$\overset{\circ}{\overbrace{E'}} \longrightarrow (\overset{\circ}{E})'\,, \quad (x',y') \longmapsto \widetilde{(x',y')}$$

is an isometry of involutive complex Banach spaces and the identity map $E' \times E' \to \overset{\circ}{\overbrace{E'}}$ is an isomorphism of real Banach spaces.

c) *Take $(x',y') \in \overset{\circ}{\overbrace{E'}}$. The following are equivalent:*

c$_1$) $\widetilde{(x',y')} \in (\overset{\circ}{E})'_+$.

c$_2$) $\widetilde{(x',y')} \in \operatorname{Re}(\overset{\circ}{E})'\,, \quad x' \in E'_+$, *and* $\|\widetilde{(x',y')}\| = \|x'\|$.

c$_3$) $\widetilde{(x',y')} \in \operatorname{Re}(\overset{\circ}{E})'$ *and*

$$x'(x^*x + y^*y) + y'(y^*x - x^*y) \in \mathbb{R}_+$$

for every $(x,y) \in \overset{\circ}{E}$.

c$_4$) $\alpha \in [-1,1] \Rightarrow \widetilde{(x',\alpha y')} \in (\overset{\circ}{E})'_+$.

d) *If A is a subset of E'_+ with infimum 0, then 0 is the infimum of $\{\widetilde{(x',0)} \mid x' \in A\}$ in $(\overset{\circ}{E})'_+$.*

e) *Given $(x,y) \in \overset{\circ}{E}$ and $(x',y') \in \overset{\circ}{\overbrace{E'}}$,*

$$(x,y)\widetilde{(x',y')} = \widetilde{(xx' - yy', xy' + yx')}\,,$$

$$\widetilde{(x',y')}(x,y) = \widetilde{(x'x - y'y, x'y + y'x)}\,.$$

f) *Take* $x' \in E$ *and let* p *be its right (left) carrier. Then* $(p,0)$ *is the right (left) carrier of* $\widetilde{(x',0)}$.

g) *We identify* $\overset{\circ}{\widehat{E'}}$ *with* $(\overset{\circ}{E})'$ *using the isometry of b) and put*

$$\widetilde{(x'',y'')} : (\overset{\circ}{E})' \longrightarrow \mathbb{C}\,, \quad (x',y') \longmapsto x''(x') - y''(y') + i(x''(y') + y''(x'))$$

for every $(x'',y'') \in \overset{\circ}{\widehat{E''}}$. *Then:*

g_1) $(x'',y'') \in \overset{\circ}{\widehat{E''}} \Rightarrow \widetilde{(x'',y'')} \in (\overset{\circ}{E})''$.

g_2) *The map*

$$\overset{\circ}{\widehat{E''}} \longrightarrow (\overset{\circ}{E})''\,, \quad (x'',y'') \longmapsto \widetilde{(x'',y'')}$$

is an isomorphism of involutive complex vector spaces.

g_3) *We have*

$$\widetilde{(x',y')}\widetilde{(x'',y'')} = \widetilde{\widetilde{(x'x'' - y'y'', x'y'' + y'x'')}},$$

$$\widetilde{(x'',y'')}\widetilde{(x',y')} = \widetilde{\widetilde{(x''x' - y''y', x''y' + y''x')}}$$

for all $(x',y') \in \overset{\circ}{\widehat{E'}}$ *and* $(x'',y'') \in \overset{\circ}{\widehat{E''}}$ *(Definition 2.2.7.8).*

a) By Proposition 4.3.6.1 d),

$$\|\widetilde{(x',y')}\| = \|\widetilde{(x',0)} + i\widetilde{(y',0)}\| \leq \|\widetilde{(x',0)}\| + \|\widetilde{(y',0)}\| = \|x'\| + \|y'\|\,.$$

Let $x \in E^{\#}$. Then $(x,0) \in \overset{\circ}{E^{\#}}$, so

$$\|\widetilde{(x',y')}\| \geq |\widetilde{(x',y')}((x,0))| = |x'(x) + iy'(x)| \geq \sup\{|x'(x)|, |y'(x)|\}\,.$$

Since x is arbitrary,

$$\|\widetilde{(x',y')}\| \geq \sup\{\|x'\|, \|y'\|\}\,.$$

b) By Proposition 4.3.6.1 f) and Proposition 2.3.1.39, the map

$$\overset{\circ}{\widehat{E'}} \longrightarrow (\overset{\circ}{E})'\,, \quad (x',y') \longmapsto \widetilde{(x',y')}$$

is involutive, linear and bijective. The assertion now follows from a).

c) Assume that $\widetilde{(x', y')} \in \operatorname{Re}(\overset{\circ}{E})'$. Then $x'^* = x'$, $y'^* = -y'$ and for any $(x, y) \in \overset{\circ}{E}$,

$$\widetilde{(x', y')}((x, y)^*(x, y)) = \widetilde{(x', y')}((x^*, -y^*)(x, y)) =$$

$$= \widetilde{(x', y')}((x^*x + y^*y, x^*y - y^*x)) =$$

$$= x'(x^*x + y^*y) + y'(y^*x - x^*y) + i(x'(x^*y - y^*x) + y'(x^*x + y^*y)) =$$

$$= x'(x^*x + y^*y) + y'(y^*x - x^*y)\,.$$

This proves the equivalence of c_1) and c_3) as well as $c_1) \Rightarrow x' \in E'_+$.

Let $\mathfrak{F}$ be an approximate unit of E (Theorem 4.2.8.2). By Proposition 4.1.2.27 i) and Proposition 4.1.1.27 d),

$$x' \in E'_+ \Longleftrightarrow x' \in \operatorname{Re} E' \text{ and } \|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x)\,,$$

$$\widetilde{(x', y')} \in (\overset{\circ}{E})'_+ \Longleftrightarrow \left\|\widetilde{(x', y')}\right\| = \lim_{x,\mathfrak{F}} \widetilde{(x', y')}((x^*x, 0))\,.$$

Moreover,

$$\lim_{x,\mathfrak{F}} \widetilde{(x', y')}((x^*x, 0)) = \lim_{x,\mathfrak{F}} (x'(x^*x) + iy'(x^*x)) = \lim_{x,\mathfrak{F}} x'(x^*x)\,.$$

Hence if $\widetilde{(x', y')} \in (\overset{\circ}{E})'_+$, then $x' \in E'_+$ and

$$\|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x) = \lim_{x,\mathfrak{F}} \widetilde{(x', y')}((x^*x, 0)) = \left\|\widetilde{(x', y')}\right\|\,.$$

If $x' \in E'_+$ and $\left\|\widetilde{(x', y')}\right\| = \|x'\|$, then

$$\left\|\widetilde{(x', y')}\right\| = \|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x) = \lim_{x,\mathfrak{F}} \widetilde{(x', y')}((x^*x, 0))$$

and we get $\widetilde{(x', y')} \in (\overset{\circ}{E})'_+$.

$c_1 \Rightarrow c_4$. Take $(x, y) \in \overset{\circ}{E}_+$. By Proposition 4.2.2.15 c), $(x, \alpha y) \in \overset{\circ}{E}_+$. By Corollary 4.2.2.10,

$$\langle (x, y)\,, \widetilde{(x', \alpha y')} \rangle = x'(x) - \alpha y'(y) = \langle (x, \alpha y), \widetilde{(x', y')} \rangle \in \mathbb{R}_+\,,$$

$$\widetilde{(x', \alpha y')} \in (\overset{\circ}{E})'_+ .$$

$c_4 \Rightarrow c_1$ is trivial.

d) Take $(x', y') \in (\overset{\circ}{E})'$ such that $\widetilde{(x', y')}$ belongs to $(\overset{\circ}{E})'_+$ and is a lower bound for $\left\{\widetilde{(z', 0)} \mid z' \in A\right\}$. By $c_1 \Rightarrow c_2$,

$$0 \leq x' \leq z'$$

for every $z' \in A$. Hence $x' = 0$ and by $c_1 \Rightarrow c_2$, $y' = 0$.

e) Take $(u, v) \in \overset{\circ}{E}$. Then

$$\left\langle (u, v), (x, y)\widetilde{(x', y')} \right\rangle = \left\langle (u, v)(x, y), \widetilde{(x', y')} \right\rangle =$$

$$= \left\langle (ux - vy, uy + vx), \widetilde{(x', y')} \right\rangle =$$

$$= x'(ux - vy) - y'(uy + vx) + i(x'(uy + vx) + y'(ux - vy)) =$$

$$= \langle u, xx' \rangle - \langle v, yx' \rangle - \langle u, yy' \rangle - \langle v, xy' \rangle +$$

$$+ i(\langle u, yx' \rangle + \langle v, xx' \rangle + \langle u, xy' \rangle - \langle v, yy' \rangle) =$$

$$= \langle u, xx' - yy' \rangle - \langle v, xy' + yx' \rangle + i(\langle u, xy' + yx' \rangle + \langle v, xx' - yy' \rangle) =$$

$$= \left\langle (u, v), \widetilde{(xx' - yy', xy' + yx')} \right\rangle ,$$

which proves the first equality. The second one can be proved similarly.

f) By e),

$$\widetilde{(x', 0)}(p, 0) = \widetilde{(x'p, 0)} = \widetilde{(x', 0)}$$

$$\left((p, 0)\widetilde{(x', 0)} = \widetilde{(px', 0)} = \widetilde{(x', 0)}\right) .$$

Take $(x, y) \in \overset{\circ}{E}$ with

$$\widetilde{(x', 0)}(x, y) = 0 \qquad \left((x, y)\widetilde{(x', 0)} = 0\right) .$$

Then by e),

$$\widetilde{(x'x, x'y)} = \widetilde{(x', 0)}(x, y) = 0 \quad \left(\widetilde{(xx', yx')} = (x, y)\widetilde{(x', 0)} = 0 \right),$$

and so, by b),

$$x'x = x'y = 0 \qquad (xx' = yx' = 0).$$

We get

$$(p, 0)(x, y) = (px, py) = 0 \qquad ((x, y)(p, 0) = (xp, yp) = 0).$$

Hence $(p, 0)$ is the right (left) carrier of $\widetilde{(x', 0)}$.

g_1) and g_2) follow from b) and Proposition 2.3.1.39.

g_3) can be proved similary to e). ■

Corollary 4.3.6.3 (0) *Let E be a real C^*-algebra. Take $x' \in E'$ and $x, y \in \mathrm{No}\, E$, such that*

$$xx' = x'y.$$

Then

$$x^*x' = x'y^*.$$

By Corollary 4.3.6.2 e),

$$(x, 0)\widetilde{(x', 0)} = \widetilde{(xx', 0)} = \widetilde{(x'y, 0)} = \widetilde{(x', 0)}(y, 0).$$

Hence, using the same argument and the Fuglede–Putnam Theorem (Theorem 4.1.4.1),

$$\widetilde{(x^*x', 0)} = (x^*, 0)\widetilde{(x', 0)} = (x, 0)^*\widetilde{(x', 0)} =$$

$$= \widetilde{(x', 0)}(y, 0)^* = \widetilde{(x', 0)}(y^*, 0) = \widetilde{(x'y^*, 0)}.$$

By Corollary 4.3.6.2 b),

$$x^*x' = x'y^*.$$ ■

Theorem 4.3.6.4 (0) *Let E be a real C^*-algebra, F an involutive E-submodule of E', and (using the notation of Corollary 4.3.6.2), take*

$$\widetilde{F} := \left\{ \widetilde{(x', y')} \mid (x', y') \in \overset{\circ}{F} \right\}.$$

a) *$\widetilde{F}$ is an involutive $\overset{\circ}{E}$-submodule of $(\overset{\circ}{E})'$.*

b) *If the map*

$$E \longrightarrow F', \quad x \longmapsto \langle x, \cdot\rangle|F$$

is an isometry, then $E \times \{0\}$ is a closed set of $\overset{\circ}{E}_{\widetilde{F}}$ and the map

$$\overset{\circ}{E} \longrightarrow (\widetilde{F})', \quad z \longmapsto \langle z, \cdot\rangle|\widetilde{F}$$

is an isometry of involutive complex Banach spaces.

a) follows from Corollary 4.3.6.2 b),e).

b) Let $\mathfrak{F}$ be a filter on $E \times \{0\}$ converging in $\overset{\circ}{E}_{\widetilde{F}}$ to (x_0, y_0). Then for $a \in F$,

$$\langle x_0, a\rangle + i\langle y_0, a\rangle = \langle (x_0, y_0), (a, 0)\rangle = \lim_{x,\mathfrak{F}} \langle (x, 0), (a, 0)\rangle = \lim_{x,\mathfrak{F}} \langle x, a\rangle \in \mathbb{R}.$$

Hence $\langle y_0, a\rangle = 0$ and $y_0 = 0$. Therefore $E \times \{0\}$ is a closed set of $\overset{\circ}{E}_{\widetilde{F}}$.

Define

$$p : \overset{\circ}{E} \longrightarrow \mathbb{R}_+, \quad z \longmapsto \|\langle z, \cdot\rangle|\widetilde{F}\|.$$

Then p is a norm on $\overset{\circ}{E}$ smaller than the initial norm of $\overset{\circ}{E}$ such that

$$p(z^*) = p(z)$$

for every $z \in \overset{\circ}{E}$.

Take $c \in \widetilde{F}^{\#}$ and $z \in \overset{\circ}{E}$ with $p(z) \leq 1$. For all $z_0 \in \overset{\circ}{E}^{\#}$, we have $z_0 c \in \widetilde{F}^{\#}$, so

$$|\langle cz, z_0\rangle| = |\langle z_0 c, z\rangle| \leq 1$$

and therefore $cz \in \widetilde{F}^{\#}$.

Take $z_1, z_2 \in \overset{\circ}{E}$ with $p(z_1) \leq 1$, $p(z_2) \leq 1$. Then, by the above consideration, we have for $c \in \widetilde{F}$,

$$|\langle c, z_1 z_2\rangle| = |\langle cz_1, z_2\rangle| \leq 1,$$

i.e. $p(z_1 z_2) \leq 1$.

Hence $\overset{\circ}{E}$ is an involutive Banach algebra with respect to p. We denote this by $\overset{\circ}{E}_p$. The map $\overset{\circ}{E}_p \to \overset{\circ}{E}$, $x \mapsto x$, being an involutive algebra homomorphism, it follows from Corollary 4.1.1.20, that $\|\cdot\| \leq p$. Hence $\|\cdot\| = p$ and the map

$$\overset{\circ}{E} \longrightarrow (\widetilde{F})', \quad z \longmapsto \langle z, \cdot\rangle|\widetilde{F}$$

is an isometry of involutive complex Banach spaces. ■

Example 4.3.6.5 *If we set*

$$u := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad v := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

then

$$\|u\| = \|v\| = \|u^*u + v^*v\|^{\frac{1}{2}} = \|(u,v)\|$$

(Proposition 2.3.1.40 and Example 4.1.1.7).

We have (Example 1.2.2.8)

$$\|u\|^2 = \frac{1}{2}\left(1+1+1+1+\sqrt{(1+1+1+1)^2 - 4|1-1|^2}\right) = 4\,,$$

$$\|v\|^2 = \frac{1}{2}\left(1+1+1+1+\sqrt{(1+1+1+1)^2 - 4|1-1|^2}\right) = 4\,,$$

$$u^*u + v^*v = 2(u+v) = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\|u^*u + v^*v\| = 4\,,$$

$$\|(u,v)\|^2 =$$

$$= \frac{1}{2}\left(2+2+2+2+\sqrt{(2+2+2+2)^2 - 4|(1+i)^2 - (1-i)^2|^2}\right) = 4\,. \quad \blacksquare$$

Remark. This example shows that the inequality of Proposition 4.3.6.1 c) is sharp (see also Example 4.1.1.28).

Proposition 4.3.6.6 (0) *Let E, F be real C^*-algebras, $u : E \to F$ an involutive algebra homomorphism, and consider*

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad (x,y) \longmapsto (ux, uy)\,.$$

If we identify $(\overset{\circ}{E})'$ with $\overbrace{E'}^{\circ}$ and $(\overset{\circ}{F})'$ with $\overbrace{F'}^{\circ}$ as in Corollary 4.3.6.2 b), then

$$(\overset{\circ}{u})'((x',y')) = (u'x', u'y')$$

for any $(x',y') \in \overset{\circ}{F}$.

By Proposition 2.3.1.41, $\overset{\circ}{u}$ is an involutive algebra homomorphism and therefore continuous (Corollary 4.1.1.20). For $(x, y) \in \overset{\circ}{E}$,

$$\langle (x,y), (\overset{\circ}{u})'((x',y')) \rangle = \langle \overset{\circ}{u}((x,y)), (x',y') \rangle = \langle (ux, uy), (x',y') \rangle =$$

$$= \langle ux, x' \rangle - \langle uy, y' \rangle + i(\langle ux, y' \rangle + \langle uy, x' \rangle) =$$

$$= \langle x, u'x' \rangle - \langle y, u'y' \rangle + i(\langle x, u'y' \rangle + \langle y, u'x' \rangle) = \langle (x,y), (u'x', u'y') \rangle ,$$

so that

$$(\overset{\circ}{u})'((x',y')) = (u'x', u'y') . \qquad \blacksquare$$

4.4 W^*–Algebras

W^*–algebras are C^*–algebras E which are duals of involutive E–submodules of E'. They represent the highpoint of the theory of involutive Banach algebras. They are order complete, which has important consequences as we have seen in the previous section. Moreover, every element of a W^*–algebra (and every element of its predual) has a polar representation and for every self–normal element x, the functional calculus may be constructed with respect to the elements of the bidual of $\mathcal{C}(\sigma(x))$ (this construction will be carried out in Subsection 6.3.3). But their most important feature is the fact that the order continuous positive linear forms are exactly the positive elements of the predual.

Throughout this section, E is always a complex (resp. real) C^*–algebra and F is always a closed vector subspace of E' (resp. an involutive E–submodule of E') such that the map

$$E \longrightarrow F', \quad x \longmapsto \langle x, \cdot \rangle | F$$

is an isometry. This is precisely the definition of a W^*–algebra (see Definition 4.4.1.1).

A great part of the results in this section were proved by S. Sakai.

4.4.1 General Properties

Definition 4.4.1.1 (0) (Sakai, 1956) *A **complex (real) W^*–algebra** is a complex (real) C^*–algebra E for which there is a closed vector subspace (an involutive E–submodule) F of E' such that the map*

$$E \longrightarrow F', \quad x \longmapsto \langle x, \cdot \rangle | F$$

*is an isometry. F is called **the predual of** E.*

Each finite–dimensional C^*–algebra is a W^*–algebra. In particular, each $\mathcal{L}(\mathbb{K}^n)$ ($n \in \mathbb{N}$) is a W^*–algebra (Example 4.1.1.7). $\ell^\infty(T)$ is a W^*–algebra for every set T (Example 1.2.2.3 d)).

We shall show (Corollary 4.4.4.4) that F appearing in the above definition is unique.

Theorem 4.4.1.2 (0) *Assume $\mathbb{K} = \mathbb{R}$. For $(a, b) \in \overset{\circ}{F}$, define*

$$\widetilde{(a,b)} : \overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x,y) \longmapsto \langle x,a\rangle - \langle y,b\rangle + i(\langle y,a\rangle + \langle x,b\rangle)$$

and put

$$\widetilde{F} := \{\widetilde{(a,b)} \mid (a,b) \in \overset{\circ}{F}\}.$$

Then $\widetilde{F}$ *is an involutive* $\overset{\circ}{E}$*-submodule of* $(\overset{\circ}{E})'$, $E \times \{0\}$ *is a closed set of* $\overset{\circ}{E}_{\widetilde{F}}$, *and the map*

$$\overset{\circ}{E} \longrightarrow (\widetilde{F})', \quad z \longmapsto \langle z, \cdot\rangle|\widetilde{F}$$

is an isometry of involutive complex Banach spaces. In particular, $\overset{\circ}{E}$ *is a complex* W^**-algebra with* $\widetilde{F}$ *as predual.*

The assertion follows from Theorem 4.3.6.4. ■

Proposition 4.4.1.3 (0) $\operatorname{Re} E$ *and* E_+ *are closed sets of* E_F, $\operatorname{Re} E^{\#}$ *and* $E_+^{\#}$ *are compact sets of* E_F, F *is an involutive set of* E', *and the map*

$$E_F \longrightarrow E_F, \quad x \longmapsto x^*$$

is continuous.

First assume $\mathbb{K} = \mathbb{R}$. Take $a \in F$. Then

$$\langle x^*, a\rangle = \langle x, a^*\rangle$$

for every $x \in E$, so that the map

$$E_F \longrightarrow E_F, \quad x \longmapsto x^*$$

is continuous. It follows that $\operatorname{Re} E$ is closed in E_F.

Step 1 $\operatorname{Re} E$ is a closed set of E_F

By the above, we may assume $\mathbb{K} = \mathbb{C}$.

Let x be a point of adherence of $\operatorname{Re} E^{\#}$ in E_F. Let $\mathfrak{F}$ be the trace of the neighbourhood filter of x in E_F on $\operatorname{Re} E^{\#}$. Take $\alpha \in \sigma(\operatorname{im} x)$. Then

$$\alpha + \beta \in \sigma(\beta 1 + \operatorname{im} x)$$

and

$$\|y + i\beta 1\|^2 = \|(y - i\beta 1)(y + i\beta 1)\| = \|y^2 + \beta^2 1\| \leq \|y\|^2 + \beta^2 \leq 1 + \beta^2$$

for every $y \in \operatorname{Re} E^{\#}$ and $\beta \in \mathbb{R}$. Hence

$$|\alpha + \beta| \leq r(\beta 1 + \operatorname{im} x) = \|\beta 1 + \operatorname{im} x\| \leq$$

$$\leq \|x + i\beta 1\| \leq \liminf_{y,\mathfrak{F}} \|y + i\beta 1\| \leq \sqrt{1 + \beta^2}$$

for every $\beta \in \mathbb{R}$ (Theorem 4.1.1.16 b), Proposition 2.3.2.7, Proposition 1.2.6.6). We deduce successively that

$$\alpha^2 + 2\alpha\beta + \beta^2 \leq 1 + \beta^2 ,$$

$$\alpha^2 + 2\alpha\beta \leq 1 ,$$

$$\alpha = 0 , \quad \operatorname{im} x = 0 , \quad x \in \operatorname{Re} E , \quad x \in \operatorname{Re} E^{\#} ,$$

(Theorem 4.1.1.16 c), Proposition 1.2.6.6). Thus $\operatorname{Re} E^{\#}$ is a closed set of E_F. By Corollary 1.3.7.5, $\operatorname{Re} E$ is a closed set of E_F.

Step 2 E_+ is a closed set of E_F

It follows from

$$E_+^{\#} \subset \operatorname{Re} E^{\#} \cap (1 + \operatorname{Re} E^{\#}) \subset E_+^{\#}$$

(Proposition 4.2.1.16 b $\Leftrightarrow$ a), that

$$E_+^{\#} = \operatorname{Re} E^{\#} \cap (1 + \operatorname{Re} E^{\#}) .$$

By Step 1, $E_+^{\#}$ is a closed set of E_F. Hence by Corollary 1.3.7.5, E_+ is a closed set of E_F.

Step 3 $\operatorname{Re} E^{\#}$ and $E_+^{\#}$ are compact sets of E_F

The assertion follows from the preceding steps and the Alaoglu–Bourbaki Theorem.

Step 4
$$\begin{cases} \text{If } \mathbb{K} = \mathbb{C} \text{ then the map} \\ E_F^{\#} \to E_F , \quad x \mapsto \operatorname{re} x \, (\operatorname{im} x) \\ \text{is continuous at } 0 . \end{cases}$$

Let $\mathfrak{F}$ be an ultrafilter on $E_F^{\#}$ converging to 0. Put

$$x := \lim_{n \to \infty} \operatorname{re} (\mathfrak{F}) , \quad y := \lim_{n \to \infty} \operatorname{im} (\mathfrak{F})$$

(Alaoglu–Bourbaki Theorem). Then

$$x + iy = \lim_{n\to\infty} \mathfrak{F} = 0\,.$$

By Step 1, $x, y \in \operatorname{Re} E$. Thus

$$x = y = 0\,,$$

which proves the assertion.

Step 5 The map $E_F \to E_F\,, \quad x \mapsto x^*$ is continuous

We may assume $\mathbb{K} = \mathbb{C}$. By Step 4, the map

$$E_F^{\#} \longrightarrow E_F\,, \quad x \longmapsto x^*$$

is continuous at 0. By Proposition 1.3.7.11 b $\Rightarrow$ a, the map

$$E_F \longrightarrow E_F\,, \quad x \longmapsto x^*$$

is continuous.

Step 6 F is an involutive set of E'

Take $a \in F$. Given $x \in E$,

$$\langle x, a^* \rangle = \overline{\langle x^*, a \rangle}$$

for every $x \in E$. By Step 5, the map

$$E_F \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle x, a^* \rangle$$

is continuous. By Proposition 1.3.7.7 c $\Rightarrow$ a, $a^* \in F$. Hence F is an involutive set of E'. ■

Remark. F is an involutive Banach space and an ordered Banach space.

Corollary 4.4.1.4 (0) *The map*

$$E \longrightarrow F'\,, \quad x \longmapsto \langle x, \cdot \rangle | F$$

is an isometry of involutive Banach spaces. ■

Corollary 4.4.1.5 (0) *If $x \in (\operatorname{Re} E) \backslash E_+$, then there is an $a \in F_+$ with*

$$\langle x, a \rangle < 0\,.$$

By Proposition 4.4.1.3 and Proposition 2.3.2.25, there is a $b \in \operatorname{Re} F$ such that

$$\sup_{y \in E_+} \langle b, y \rangle = 0 < \langle b, x \rangle .$$

Put

$$a := -b .$$

Then

$$\langle y, a \rangle = -\langle y, b \rangle \geq 0$$

for every $y \in E_+$. By Corollary 4.2.2.10, a is positive. Hence $a \in F_+$ and

$$\langle x, a \rangle = -\langle x, b \rangle < 0 .$$ ■

Corollary 4.4.1.6 (0) *Let* $\mathbb{K} = \mathbb{R}$ $(\mathbb{K} = \mathbb{C})$ *and* $x \in \operatorname{Re} E$ $(x \in E)$.

a) x *is positive iff*

$$\langle x, a \rangle \in \mathbb{R}_+$$

for every $a \in F_+$.

b) $x = 0$ *iff* $\langle x, a \rangle = 0$ *for all* $a \in F_+$.

a) If x is positive, then

$$\langle x, a \rangle \in \mathbb{R}_+$$

for every $a \in F_+$ (Theorem 4.2.2.1 a $\Rightarrow$ d).

Now suppose that

$$\langle x, a \rangle \in \mathbb{R}_+$$

for every $a \in F_+$. Then

$$\langle \operatorname{re} x, a \rangle = \operatorname{re} \langle x, a \rangle \in \mathbb{R}_+$$

$$(\langle \operatorname{im} x, a \rangle = \operatorname{im} \langle x, a \rangle = 0)$$

for every $a \in E_+$ (Corollary 2.3.1.23 a $\Rightarrow$ c) and so

$$\operatorname{re} x \in E_+ \quad (\operatorname{im} x = 0)$$

(Corollary 4.4.1.5). Hence $x \in E_+$.

b) follows from a). ■

Corollary 4.4.1.7 *E'_+ is the closure of F_+ in E'_E.*

Let A' be the closure of F_+ in E'_E. It is obviously a convex cone contained in E'_+. Assume there is an $x' \in E'_+ \backslash A'$. By Proposition 2.3.2.25, there is an $x \in \operatorname{Re} E$ with

$$\sup_{y' \in A'} y'(x) = 0 < x'(x) .$$

Then $-x \in E_+$, by Corollary 4.4.1.5). Hence

$$0 \leq x'(-x) = -x'(x) < 0$$

by Theorem 4.2.2.1 a $\Rightarrow$ d. The contradiction obtained implies that $E'_+ = A'$. ∎

Theorem 4.4.1.8 (0)

a) *F is order complete and an order faithful set of E'.*

b) *Every norm bounded upward directed nonempty set A of E_+ has a supremum x and the upper section filter of A converges to x in E_F.*

c) *E is $\mathbb{C}$-order complete and a is order continuous for every $a \in F$.*

d) *Every subset of E which is closed in E_F is order faithful and order complete. Every involutive subalgebra of E which is closed in E_F is a W^*-algebra.*

e) *E_+, $\operatorname{Re} E^\#$, and $E^\#_+$ are order complete and order faithful sets of E.*

f) *For every $x \in \operatorname{Sn} E$ and $\varepsilon > 0$, there is a finite family $((\alpha_\iota, p_\iota))_{\iota \in I}$ in $\sigma(x) \times \operatorname{Pr} E$ such that*

$$\left\| x - \sum_{\iota \in I} \alpha_\iota p_\iota \right\| < \varepsilon .$$

g) *If $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$), then the vector subspace of E generated by $\operatorname{Pr} E$ is dense in E (in $\operatorname{Re} E$.).*

h) *E has a unit.*

i) *$\operatorname{Pr} E$ is an order complete lattice and an order faithful set of E.*

j) *Every element of E has a left and a right carrier.*

k) *Every maximal Gelfand C^*–subalgebra of E is order complete.*

l) *For every $x \in \operatorname{Re} E$ $(x \in E_+)$, there is an increasing sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to its supremum x such that for every $n \in \mathbb{N}$ there is a finite family $(\alpha_\iota, p_\iota)_{\iota\in I}$ in $\mathbb{R} \times \operatorname{Pr} E$ (in $\mathbb{R}_+ \times \operatorname{Pr} E$) with*

$$x_n = \sum_{\iota\in I} \alpha_\iota p_\iota .$$

a) follows from Proposition 1.7.2.5.

b) follows from Proposition 4.4.1.3 and Proposition 1.7.2.6.

c) By Corollary 4.2.1.18, every upper bounded set of E_+ is bounded in norm and so the assertion follows from b) and Theorem 4.4.1.2.

d) The first assertion follows from b) and c). In order to prove the second assertion, let G be an involutive subalgebra of E which is closed in E_F. By Theorem 1.3.5.12 c), G' may be identified with E'/G°. $F/F \cap G^\circ$ is a closed vector subspace of E'/G° and, in the real case, it is a G–submodule of E'/G°. By Corollary 1.3.5.5, the map

$$G \longrightarrow (F/F \cap G^\circ)', \quad x \longmapsto \langle x, \cdot\rangle|(F/F \cap G^\circ)$$

is an isometry. Hence G is a W^*–algebra.

e) follows from d) and Proposition 4.4.1.3.

f) and g) follow from c) and Corollary 4.3.2.4.

h) By Theorem 4.4.1.2, we may assume $\mathbb{K} = \mathbb{C}$. By c) and Corollary 4.3.2.6, $\operatorname{Pr} E$ is upward directed. By b), $\operatorname{Pr} E$ has a supremum x in $E_+^\#$. Then $p \leq x$ and so

$$px = xp = p$$

for every $p \in \operatorname{Pr} E$ (Corollary 4.2.7.3 a $\Rightarrow$ d). By g), x is a unit of E.

i) follows from c) and Proposition 4.2.7.17.

j) By c) and h), E is unital and order complete. Thus by Corollary 4.3.3.11, every element of E has a right and a left carrier.

k) follows from c), Theorem 4.4.1.2, and Corollary 4.2.2.20 a).

l) follows from c),f), and Corollary 4.3.2.11. ∎

Corollary 4.4.1.9 *$F_+^\#$ is dense in $({E'_+}^\#)_E$ and $\{a \in F_+ \mid \|a\| = 1\}$ is dense in $\tau(E)$.*

Take $x' \in E'_+ \backslash \{0\}$. By Corollary 4.4.1.7, there is a filter $\mathfrak{F}$ on F_+, such that $j_F(\mathfrak{F})$ converges to x' in E'_E. Then

$$\lim_{a,\mathfrak{F}} \|a\| = \lim_{a,\mathfrak{F}} \langle 1, a \rangle = \langle 1, x' \rangle = \|x'\|$$

(Theorem 4.4.1.8 h), Corollary 2.3.4.7). Hence

$$\lim_{a,\mathfrak{F}} \frac{1}{\|a\|} a = \frac{1}{\|x'\|} x'$$

in E'_E. ■

Corollary 4.4.1.10 (0) *If E is Gelfand, then $\sigma(E)$ is a compact hyper-Stonian space.*

By Theorem 4.4.1.8 h), E has a unit, $\sigma(E)$ is thus compact (Theorem 2.4.1.3 c)). By Theorem 4.4.1.8 c), E is order complete and so $\sigma(E)$ is a Stone space (Proposition 1.7.2.13 d $\Rightarrow$ a). Let U be a nonempty open set of $\sigma(E)$. There is an $x \in \operatorname{Re} E \backslash \{0\}$ with

$$\operatorname{Supp} \widehat{x} \subset U$$

(Urysohn's Theorem). By Corollary 4.4.1.6 b), there is an $a \in F$ with

$$\langle x, a \rangle \neq 0 .$$

Let μ be the Radon measure on $\sigma(E)$ for which

$$\int \widehat{y} d\mu = \langle y, a \rangle$$

for every $y \in E$. Then

$$\int \widehat{x} d\mu = \langle x, a \rangle \neq 0 .$$

Thus

$$U \cap \operatorname{Supp} \mu \neq \emptyset .$$

By Theorem 4.4.1.8 c), $a \in E^\pi$ and so $\mu \in C(T)^\pi$. Hence T is a hyper-Stonian space. ■

Corollary 4.4.1.11 (0) *Take $x \in \operatorname{Sn} E$ and let $j : \sigma(x) \to \mathbb{K}$ be the inclusion map. Let $\mathfrak{B}$ be the σ-algebra of Borel sets of $\sigma(x)$ and $\mathcal{B}$ the C^*-algebra of bounded Borel functions on $\sigma(x)$.*

a) *There is a unique E-valued spectral measure μ on $\mathfrak{B}$ such that*

$$\int j d\mu = x$$

(in fact μ is the spectral measure of x from Theorem 4.3.2.19).

b) *For every $f \in \mathcal{B}$ and every order σ-faithful C^*-subalgebra G of E containing x,*

$$f(x) = \int f d\mu\,, \quad \sigma(f(x)) \subset \overline{f(\sigma(x))}\,,$$

$$\|f(x)\| \leq \|f\|\,, \quad \{x\}^c \subset \{f(x)\}^c\,, \quad f(x) \in G\,,$$

and

$$\left\{ \widehat{\overset{G}{f(x)}} \neq f \circ \overset{G}{\widehat{x}} \right\}$$

is meager whenever G is a Gelfand C^*-algebra.

c) *If $f \in \mathcal{B}$ and g a bounded Borel function on $\overline{f(\sigma(x))}$, then*

$$g(f(x)) = g \circ f(x)\,.$$

d) $\mu(\mathfrak{B}) \subset \mu(\mathfrak{B})^c = \{x\}^c$.

e) *$a \circ \mu$ is a measure for every $a \in F$ and*

$$\int f d(a \circ \mu) = \left\langle \int f d\mu, a \right\rangle$$

for every $f \in \mathcal{B}$.

f) *x is atomic iff $a \circ \mu$ is atomic for every $a \in F$.*

g) *If 0 is a nonisolated point of $\sigma(x)$, then there is a sequence $(p_n)_{n\in\mathbb{N}}$ in $\{x\}^c \cap \operatorname{Pr} E \backslash \{0\}$ such that*

$$p_m p_n = 0$$

for all distinct $m, n \in \mathbb{N}$ and

$$x = \sum_{n\in\mathbb{N}} x p_n\,.$$

By Theorem 4.4.1.8 c),h), E is unital and $\mathbb{C}$-order complete. Thus the assertions follow from Theorem 4.3.2.19, Corollary 4.3.2.5, and Corollary 4.4.1.6. ■

Corollary 4.4.1.12 *Let A be a subset of $\mathbb{R}$ and f an increasing Borel function on A. Take $x, y \in \operatorname{Re} E$ with*

$$\sigma(x) \subset A\,, \quad \sigma(y) \subset A\,, \quad x \leq y\,, \quad xy = yx\,.$$

Then

$$f(x) \leq f(y)\,.$$

Let G be a maximal Gelfand C^*-subalgebra of E containing x and y (Corollary 4.1.2.3). Then

$$\overset{G}{\widehat{x}} \leq \overset{G}{\widehat{y}}$$

(Proposition 4.2.1.9) and so

$$f \circ \overset{G}{\widehat{x}} \leq f \circ \overset{G}{\widehat{y}}\,.$$

Since

$$\left\{\widehat{\overset{G}{f(x)}} \neq f \circ \overset{G}{\widehat{x}}\right\}\,, \quad \left\{\widehat{\overset{G}{f(y)}} \neq f \circ \overset{G}{\widehat{y}}\right\}$$

are meager (Corollary 4.3.2.5 c)), it follows that

$$\left\{\widehat{\overset{G}{f(y)}} < \widehat{\overset{G}{f(x)}}\right\}$$

is meager. Since $\sigma(G)$ is a Baire space,

$$\widehat{\overset{G}{f(x)}} \leq \widehat{\overset{G}{f(y)}}\,,$$

$$f(x) \leq f(y)$$

(Proposition 4.2.1.9). ■

Corollary 4.4.1.13 *If C is a hereditary cone (hereditary C^*-subalgebra) of E which is closed in E_F, then, given any $x \in C \cap E_+$, ($x \in C \cap \operatorname{Re} E$), the carrier of x belongs to C (Theorem 4.4.1.8 j)).*

First assume $x \in E_+$. Define

$$j : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha,$$

$$f : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0, \end{cases}$$

and for each $n \in \mathbb{N}$, define

$$f_n : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} 1 & \text{if } \alpha \geq \frac{1}{n} \\ 0 & \text{if } \alpha < \frac{1}{n}. \end{cases}$$

Then

$$f_n \leq nj, \quad f_n \leq f_{n+1} \leq e_{\sigma(x)}^{\sigma(x)}$$

so that

$$f_n(x) \leq nj(x) = nx,$$

$$f_n(x) \leq f_{n+1}(x) \leq 1,$$

$$f_n(x) \in C$$

for every $n \in \mathbb{N}$ (Corollary 4.3.2.5 a), Proposition 4.3.4.3). By Corollary 4.3.2.5 a),c),

$$f(x) = \bigvee_{n \in \mathbb{N}} f_n(x)$$

and by Corollary 4.3.3.11 (and Corollary 4.3.3.10 a $\Leftrightarrow$ b, Corollary 4.3.3.9), $f(x)$ is the carrier of x. By Theorem 4.4.1.8 b), $(f_n(x))_{n \in \mathbb{N}}$ converges to $f(x)$ in E_F. Thus $f(x) \in C$.

Now assume $x \in \operatorname{Re} E$ and let p and q be the carriers of x^+ and x^-, respectively (Theorem 4.4.1.8 j)). By the above, $p, q \in C$. By Proposition 4.3.3.15, $p + q$ is the carrier of x. Hence the carrier of x belongs to C. ∎

Corollary 4.4.1.14 *Let $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in F (in $\operatorname{Re} F$) such that $(\langle p, a_n \rangle)_{n \in \mathbb{N}}$ converges whenever $p \in \operatorname{Pr} E$. Then there is an order continuous $x' \in E'$ ($x' \in \operatorname{Re} E'$) such that*

$$\lim_{n \to \infty} \langle x, a_n \rangle = x'(x)$$

for every $x \in E$.

By Theorem 4.3.2.15 (and Theorem 4.4.1.8 c)), there is a $y' \in E'$ such that

$$\lim_{n\to\infty} \langle x, a_n \rangle = y'(x) \qquad (\lim_{n\to\infty} \langle x, a_n \rangle = y'(\operatorname{re} x))$$

for every $x \in E$. Put

$$x' := \begin{cases} y' & \text{if } \mathbb{K} = \mathbb{C} \\ \operatorname{re} y' & \text{if } \mathbb{K} = \mathbb{R}. \end{cases}$$

Then

$$\lim_{n\to\infty} \langle x, a_n \rangle = x'(x)$$

for every $x \in E$. By Theorem 4.4.1.8 c), a_n is order continuous for every $n \in \mathbb{N}$. Since E is order complete (Theorem 4.4.1.8 c)), it follows from Proposition 1.7.2.11, that x' is order continuous. ■

Example 4.4.1.15 *If $(T, \mathfrak{T}, \mu)$ is a measure space, then the C^*-algebra $L^\infty(\mu)$ (Example 4.1.1.5) is a W^*-algebra iff $L^\infty(\mu)$ is order complete. This condition is fulfilled whenever μ is σ-bounded or a Radon measure on a locally compact space.*

By Example 4.2.3.8, the order relation in the C^*-algebra $L^\infty(\mu)$ coincides with the usual order relation on $L^\infty(\mu)$.

If $L^\infty(\mu)$ is a W^*-algebra, then by Theorem 4.4.1.8 c), $L^\infty(\mu)$ is order complete. If $L^\infty(\mu)$ is order complete, then it is isometric to the dual of $L^1(\mu)$. Thus $L^\infty(\mu)$ is a W^*-algebra. The proof of the last assertion is standard and may be found in most books on integration theory. ■

4.4.2 F as an E–submodule of E'

Proposition 4.4.2.1 (0) *$pE(1-p)+(1-p)Ep$ is a closed set of E_F for every $p \in \Pr E$.*

First assume $\mathbb{K} = \mathbb{R}$. Take $a \in F$. Then

$$\langle px(1-p)+(1-p)xp, a\rangle = \langle x, (1-p)ap + pa(1-p)\rangle$$

for every $x \in E$, so that the map

$$E_F \longrightarrow E_F, \quad x \longmapsto px(1-p)+(1-p)xp$$

is continuous. Take $x \in pE(1-p)+(1-p)Ep$. Then there are $y, z \in F$ such that

$$x = py(1-p)+(1-p)zp.$$

It follows

$$px(1-p)+(1-p)xp = py(1-p)+(1-p)zp = x.$$

Hence

$$pE(1-p)+(1-p)Ep = \{x \in E \mid x = px(1-p)+(1-p)xp\}.$$

By the above, $pE(1-p)+(1-p)Ep$ is a closed set of E_F.

Now assume $\mathbb{K} = \mathbb{C}$. Throughout this proof, we consider E endowed with the topology of E_F. We note that pEp is a C^*–subalgebra of E for which p is a unit. We denote by G the corresponding unital C^*–algebra.

Let x be a point of adherence of $pE^{\#}(1-p)$. There is a filter $\mathfrak{F}$ on E with

$$E^{\#} \in \mathfrak{F}, \quad \lim_{y,\mathfrak{F}} py(1-p) = x.$$

Given $y \in E^{\#}$ and $\alpha \in \mathbb{C}$,

$$\|py(1-p)+\alpha p\|^2 = \|(py(1-p)+\alpha p)((1-p)y^*p+\overline{\alpha}p)\| =$$

$$= \|py(1-p)y^*p + |\alpha|^2 p\| \leq \|py(1-p)y^*p\| + |\alpha|^2\|p\| \leq 1 + |\alpha|^2,$$

so that

$$\|pxp + \alpha p\| = \|p(x + \alpha p)p\| \leq \|x + \alpha p\| \leq$$

$$\leq \liminf_{y,\mathfrak{F}} \|py(1-p) + \alpha p\| \leq \sqrt{1 + |\alpha|^2}$$

(Proposition 1.2.6.6) for every $\alpha \in \mathbb{C}$.

Take $\alpha \in \mathbb{R}$. Put

$$y := \operatorname{re} x \quad (y := \operatorname{im} x),$$

and take $\beta \in \sigma_G(pyp)$. Then $\beta \in \mathbb{R}$ (Theorem 4.1.2.1, Proposition 2.3.1.29)

$$\alpha + \beta = \sigma_G(pyp + \alpha p),$$

$$\|pyp + \alpha p\| \leq \|pxp + \alpha p\| \leq \sqrt{1 + \alpha^2}$$

$$\left(\|pyp + \alpha p\| \leq \|pxp + i\alpha p\| \leq \sqrt{1 + \alpha^2}\right)$$

(Proposition 2.3.2.7), so that

$$|\alpha + \beta| \leq r_G(pyp + \alpha p) \leq \|pyp + \alpha p\| \leq \sqrt{1 + \alpha^2}$$

(Proposition 2.2.4.1). Since α is arbitrary, we deduce successively that

$$\beta = 0,$$

$$pyp = 0$$

(Theorem 4.1.1.16 c)),

$$pxp = 0.$$

Similarly,

$$(1-p)x(1-p) = 0.$$

Now let x be a point of adherence of

$$A := E^{\#} \cap (pE(1-p) + (1-p)Ep).$$

Then there is an ultrafilter $\mathfrak{F}$ on A converging to x. Put

$$x_1 := \lim_{y,\mathfrak{F}} py(1-p), \quad x_2 := \lim_{y,\mathfrak{F}} (1-p)yp$$

(Alaoglu–Bourbaki Theorem). Then

$$x = x_1 + x_2$$

and, by the above proof,

$$pxp = px_1p + px_2p = 0\,,$$

$$(1-p)x(1-p) = (1-p)x_1(1-p) + (1-p)x_2(1-p) = 0\,.$$

Hence

$$x = ((1-p)+p)x((1-p)+p) = px(1-p) + (1-p)xp \in A\,.$$

A is therefore a closed set. By Corollary 1.3.7.5, $pE(1-p) + (1-p)Ep$ is a closed set of E_F. ■

Proposition 4.4.2.2 (0) *Given $p \in \Pr E$, the set pEp is closed in E_F and the map*

$$E_F \longrightarrow E_F\,, \quad x \longmapsto pxp$$

is continuous. pEp is an order faithful set of E and a W^*-algebra with p as unit.

It is easy to see that the map

$$E_F \longrightarrow E_F\,, \quad x \longmapsto pxp$$

is continuous and that pEp is a closed set of E_F if $\mathbb{K} = \mathbb{R}$. So assume $\mathbb{K} = \mathbb{C}$. By Proposition 4.2.7.1 c $\Leftrightarrow$ d,

$$E^{\#} \cap (p(\operatorname{Re} E)p) = \{x \in \operatorname{Re} E \mid -p \le x \le p\} = (E_+ - p) \cap (p - E_+)\,.$$

Hence $E^{\#} \cap (p(\operatorname{Re} E)p)$ is a closed set of E_F (Proposition 4.4.1.3). By Corollary 1.3.7.5, $p(\operatorname{Re} E)p$ is a closed set of E_F.

Consider

$$u : E_F \longrightarrow E_F\,, \quad x \longmapsto \operatorname{re} x\,,$$

$$v : E_F \longrightarrow E_F\,, \quad x \longmapsto \operatorname{im} x\,.$$

By Proposition 4.4.1.3, u and v are continuous. From

$$pEp = \bar{u}^{-1}(p(\operatorname{Re} E)p) \cap \bar{v}^{-1}(p(\operatorname{Re} E)p)$$

it follows that pEp is a closed set of E_F.

Consider

$$u : E \longrightarrow E\,, \quad x \longmapsto pxp\,,$$

$$v : E \longrightarrow E\,, \quad x \longmapsto (1-p)x(1-p)\,,$$

$$w : E \longrightarrow E\,, \quad x \longmapsto px(1-p) + (1-p)xp\,.$$

u, v, w are projections in E and

$$vw = wv = 0\,,$$

$$\operatorname{Im} v = (1-p)E(1-p)\,,$$

$$\operatorname{Im} w \subset pE(1-p) + (1-p)Ep$$

as can easily be seen. Take $x, y \in E$ and put

$$z := px(1-p) + (1-p)yp\,.$$

Then

$$wz = px(1-p) + (1-p)yp = z$$

and so $z \in \operatorname{Im} w$.

$$pE(1-p) + (1-p)Ep \subset \operatorname{Im} v\,,$$

$$\operatorname{Im} w = pE(1-p) + (1-p)Ep\,.$$

By the above proof and Proposition 4.4.2.1, $\operatorname{Im} v$ and $\operatorname{Im} w$ are closed sets of E_F. Hence $\operatorname{Im} v + \operatorname{Im} w$ is a closed set of E_F (Proposition 1.3.7.6). Since

$$\operatorname{Im} u = pEp\,,$$

$$\operatorname{Ker} u = \operatorname{Im} v + \operatorname{Im} w\,,$$

it follows that $\operatorname{Im} u$ and $\operatorname{Ker} u$ are closed sets of E_F. By Corollary 1.3.7.12 b $\Rightarrow$ a, the map

$$E_F \longrightarrow E_F\,, \quad x \longmapsto pxp$$

is continuous.

The last assertion follwos from Theorem 4.4.1.8 d). ■

Proposition 4.4.2.3 (0) *The map*

$$E_F \longrightarrow E_F\,, \quad x \longmapsto xp \quad (\textit{resp. } px)$$

is continuous for every $p \in \Pr E$.

Let x be a point of adherence of $E^{\#} \cap (pE(1-p))$ in E_F. By Proposition 4.4.2.1, there are $y, z \in E$ such that

$$x = py(1-p) + (1-p)zp\,.$$

Hence

$$px(1-p) + (1-p)xp = py(1-p) + (1-p)zp = x\,.$$

Assume that

$$(1-p)xp \neq 0\,.$$

Let α be a real number,

$$\alpha > \frac{1}{\|(1-p)xp\|}\,.$$

Then

$$\|x + \alpha(1-p)xp\| = \|px(1-p) + (1+\alpha)(1-p)xp\| =$$

$$= \sup\{\|px(1-p)\|\,,\, (1+\alpha)\|(1-p)xp\|\} = (1+\alpha)\|(1-p)xp\|$$

(Proposition 4.1.2.23). On the other hand,

$$\|py(1-p) + \alpha(1-p)xp\| = \sup\{\|py(1-p)\|\,,\, \alpha\|(1-p)xp\|\} \leq$$

$$\leq \sup\{1, \alpha\|(1-p)xp\|\} = \alpha\|(1-p)xp\|$$

for every $y \in E^{\#}$ (Proposition 4.1.2.23). Since $x + \alpha(1-p)xp$ is a point of adherence of

$$E^{\#} \cap (pE(1-p)) + \alpha(1-p)xp$$

in E_F, we deduce the contradiction

$$(1+\alpha)\|(1-p)xp\| \leq \alpha\|(1-p)xp\|$$

(Proposition 1.2.6.6). Hence

$$(1-p)xp = 0\,,$$

$$x = px(1-p) \in E^{\#} \cap (pE(1-p))\,,$$

and $E^{\#} \cap (pE(1-p))$ is a closed set of E_F. By Corollary 1.3.7.5, $pE(1-p)$ is a closed set of E_F.

Consider

$$u : E \longrightarrow E\,, \quad x \longmapsto pxp\,,$$

$$v : E \longrightarrow E\,, \quad x \longmapsto (1-p)x(1-p)\,,$$

$$w : E \longrightarrow E\,, \quad x \longmapsto px(1-p)\,,$$

$$t : E \longrightarrow E\,, \quad x \longmapsto (1-p)xp\,.$$

u, v, w, t are projections in E and

$$uv = vu = uw = wu = vw = wv = 0\,.$$

By the above considerations and by Proposition 4.4.2.2, the sets $\operatorname{Im} u$, $\operatorname{Im} v$, $\operatorname{Im} w$, $\operatorname{Im} t$ are closed in E_F. From

$$\operatorname{Ker} t = \operatorname{Im} u + \operatorname{Im} v + \operatorname{Im} w$$

it follows that $\operatorname{Ker} t$ is closed in E_F (Proposition 1.3.7.6). By Corollary 1.3.7.12 b $\Rightarrow$ a, the map

$$E_F \longrightarrow E_F\,, \quad x \longmapsto (1-p)xp$$

is continuous. Hence the map

$$E_F \longrightarrow E_F\,, \quad x \longmapsto px(1-p)$$

is also continuous. Since

$$xp = pxp + (1-p)xp\,, \quad px = pxp + px(1-p)$$

for every $x \in E$, we deduce that the map

$$E_F \longrightarrow E_F\,, \quad x \longmapsto xp \quad (\text{resp. } px)$$

is continuous (Proposition 4.4.2.2). ■

Theorem 4.4.2.4 (0) *Given $x \in E$, the map*

$$E_F \longrightarrow E_F, \quad y \longmapsto xy \quad (resp.\ yx)$$

is continuous.

The assertion is easy to see if $\mathbb{K} = \mathbb{R}$. So assume $\mathbb{K} = \mathbb{C}$. Let G be the set of $x \in E$ for which the map

$$E_F^{\#} \longrightarrow E_F, \quad y \longmapsto xy \quad (\text{resp. } yx)$$

is continuous. G is obviously a closed vector subspace of E. By Proposition 4.4.2.3, $\Pr E \subset G$, so that $G = E$ by Theorem 4.4.1.8 g). The assertion now follows from Proposition 1.3.7.11 b $\Rightarrow$ a. ■

Corollary 4.4.2.5 (0) *Given $A \subset E$, A^c is a closed set of E_F.*

Given $x \in E$, put

$$u_x : E_F \longrightarrow E_F, \quad y \longmapsto xy - yx.$$

By Theorem 4.4.2.4, u_x is continuous, so that $\operatorname{Ker} u_x$ is closed in E_F for every $x \in E$. The assertion now follows from

$$A^c = \bigcap_{x \in A} \operatorname{Ker} u_x. \qquad \blacksquare$$

Corollary 4.4.2.6 *Let A be a commutative set of E and x a point of adherence of A in E_F (this is the case, for example, when there is a downward directed subset B of A with x as infimum). Then $A \cup \{x\}$ is commutative.*

Since A^c is closed in E_F (Corollary 4.4.2.5) and contains A, we see that $x \in A^c$, i.e. $A \cup \{x\}$ is commutative. The assertion in parentheses follows from Theorem 4.4.1.8 b). ■

Corollary 4.4.2.7 (0) *Every maximal Gelfand C^*-subalgebra G of E is closed in E_F and so it is order faithful, order complete, and a W^*-algebra and $\sigma(G)$ is a compact hyperstonian space. If A is a downward directed set of G and x is its infimum in E, then $x \in G$ and*

$$\left\{ x' \in \sigma(G) \;\middle|\; \overset{G}{\widehat{x}}(x') < \inf_{y \in A} \overset{G}{\widehat{y}}(x') \right\}$$

is meager.

By Corollary 4.1.4.2 b),c), G^{cc} is a commutative C^*-subalgebra of E containing G and by Corollary 4.4.2.5, G^{cc} is closed in E_F. If $\mathbb{K} = \mathbb{C}$, then G^{cc} is a Gelfand C^*-subalgebra of E. If $\mathbb{K} = \mathbb{R}$, then $G^{cc} \cap \operatorname{Re} E$ is a Gelfand C^*-subalgebra of E. Moreover, in this case, the map

$$E_F \longrightarrow E_F, \quad y \longmapsto y^*$$

is continuous, so that $\operatorname{Re} E$ and $G^{cc} \cap \operatorname{Re} E$ are closed in E_F. Since G is maximal, $G = G^{cc}$ in the complex case and $G = G^{cc} \cap \operatorname{Re} E$ in the real case. Thus G is closed in E_F in both cases. By Theorem 4.4.1.8 d), G is order faithful, order complete, and a W^*-algebra. By Corollary 4.4.1.10, $\sigma(G)$ is a compact hyperstonian space. Since G is an order faithful set of E, $x \in G$ and by Proposition 1.7.2.13 a $\Rightarrow$ c,

$$\left\{ x' \in \sigma(G) \;\middle|\; \overset{G}{\widehat{x}}(x') < \inf_{y \in A} \overset{G}{\widehat{y}}(x') \right\}$$

is meager. ■

Corollary 4.4.2.8 *Let A be a commutative downward directed set of $\operatorname{Re} E$, x its infimum, and f an increasing upper semicontinuous real function on $\sigma(x) \cup \{ \bigcup_{y \in A} \sigma(y) \}$. Then $(f(y))_{y \in A}$ is a downward directed family in E with infimum $f(x)$.*

Let G be a maximal Gelfand C^*-subalgebra of E containing A (Corollary 4.1.2.3). By Corollary 4.4.2.7, $\sigma(G)$ is a compact hyperstonian space, $x \in G$, and

$$\left\{ x' \in \sigma(G) \;\middle|\; \widehat{x}(x') < \inf_{y \in A} \widehat{y}(x') \right\}$$

is a meager set of $\sigma(G)$. Since f is upper semicontinuous,

$$\left\{ x' \in \sigma(G) \;\middle|\; f \circ \widehat{x}(x') < \inf_{y \in A} f \circ \widehat{y}(x') \right\} \subset \left\{ x' \in \sigma(G) \;\middle|\; \widehat{x}(x') < \inf_{y \in A} \widehat{y}(x') \right\}$$

and

$$\widehat{f \circ y} \leq f \circ \widehat{y}$$

for every $y \in A$. Moreover,

$$\left\{ \widehat{f \circ x} \neq f \circ \widehat{x} \right\}$$

is meager. From

$$\left\{x' \in \sigma(G) \;\middle|\; \widehat{f(x)}(x') < \inf_{y \in A} \widehat{f(y)}(x')\right\} \subset$$

$$\subset \left\{\widehat{f \circ x} \neq f \circ \widehat{x}\right\} \cup \{x' \in \sigma(x) \mid f \circ \widehat{x}(x') < \inf f \circ \widehat{y}(x')\}$$

it follows that

$$\left\{x' \in \sigma(G) \;\middle|\; \widehat{f(x)}(x') < \inf_{y \in A} \widehat{f(y)}(x')\right\}$$

is a meager set of $\sigma(G)$.

Let z be the infimum of $\{f(y) \mid y \in A\}$ in E. By Corollary 4.4.1.12, $(f(y))_{y \in A}$ is downward directed. By Corollary 4.4.2.7, $z \in G$. Since $f(x) \leq z$, it follows that

$$\left\{\widehat{f(x)} \neq \widehat{z}\right\} \subset \left\{x' \in \sigma(G) \;\middle|\; \widehat{f(x)}(x') < \inf_{y \in A} \widehat{f(y)}(x')\right\}.$$

Hence

$$\left\{\widehat{f(x)} \neq \widehat{z}\right\}$$

is meager,

$$f(x) = z,$$

and $f(x)$ is the infimum of $(f(y))_{y \in A}$. ■

Remark. The above relation need not hold if f is lower semicontinuous. Let x be an invertible element of E_+ and consider

$$f : [0, \|x\|] \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} 1 & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0. \end{cases}$$

Then $\bigwedge_{n \in \mathbb{N}} \frac{1}{n} x = 0$ (Corollary 4.2.2.2) while $f(\frac{1}{n} x) = 1$ for every $n \in \mathbb{N}$.

Corollary 4.4.2.9 (0) *The closure in E_F of any involutive subalgebra of E is a C^*-subalgebra of E.*

Let G be an involutive subalgebra of E and $\overline{G}$ its closure in E_F. Consider

$$f : E_F \longrightarrow E_F, \quad x \longmapsto x^*.$$

By Proposition 4.4.1.3, f is continous. From

$$G = \overset{-1}{f}(G) \subset \overset{-1}{f}(\overline{G}),$$

we see that

$$\overline{G} \subset \overset{-1}{f}(\overline{G}),$$

i.e. $\overline{G}$ is an involutive set of E.

Take $x \in E$. Consider

$$g_1 : E_F \longrightarrow E_F, \quad y \longmapsto x + y,$$
$$g_2 : E_F \longrightarrow E_F, \quad y \longmapsto xy,$$
$$g_3 : E_F \longrightarrow E_F, \quad y \longmapsto yx.$$

g_1, g_2, and g_3 are continuous (Theorem 4.4.2.4). If $x \in G$, then

$$G \subset \bigcap_{i=1}^{3} \overset{-1}{g_i}(G) \subset \bigcap_{i=1}^{3} \overset{-1}{g_i}(\overline{G}),$$

so that

$$\overline{G} \subset \bigcap_{i=1}^{3} \overset{-1}{g_i}(\overline{G}).$$

It follows that

$$G \subset \bigcap_{i=1}^{3} \overset{-1}{g_i}(\overline{G}).$$

and so

$$\overline{G} \subset \bigcap_{i=1}^{3} \overset{-1}{g_i}(\overline{G}),$$

for every $x \in \overline{G}$. Hence $\overline{G}$ is an involutive subalgebra of E. Being closed, it is a C^*-subalgebra of E. ■

Corollary 4.4.2.10 (0) *F is an involutive E–submodule of E' and the maps*

$$E_F \longrightarrow F_E, \quad y \longmapsto ya \quad (resp. \quad ay),$$

$$F_E \longrightarrow F_E, \quad b \longmapsto xb \quad (resp. \quad bx),$$

are continuous for all $a \in F$ *and* $x \in E$. *Moreover, the map*

$$E \longrightarrow F', \quad x \longmapsto \langle x, \cdot \rangle \mid F$$

is an isometry of involutive E*-modules. In particular, for* $\mathbb{K} = \mathbb{C}$, *if we restrict the scalars of* E *to* $\mathbb{R}$, *then* E *becomes a real* W^**-algebra.*

By Theorem 4.4.2.4, the map

$$E_F \longrightarrow \mathbb{K}, \quad y \longmapsto \langle yx, a \rangle \quad (\text{resp. } \langle xy, a \rangle)$$

is continuous. Hence xa (resp. ax) belong to F (Proposition 1.3.7.7 c $\Rightarrow$ a) and the map

$$E_F \longrightarrow F_E \quad y \longmapsto ay \quad (\text{resp. } ya)$$

is continuous. By Proposition 4.4.1.3, F is an involutive E-submodule of E'. Since the map

$$F_E \longrightarrow \mathbb{K}, \quad b \longmapsto \langle yx, b \rangle \quad (\text{resp. } \langle xy, b \rangle)$$

is continuous for every $y \in E$, the map

$$F_E \longrightarrow F_E \quad b \longmapsto xb \quad (\text{resp. } bx)$$

is continuous.

By Proposition 2.2.7.5, j_E is a homomorphism of E-modules. It is easy to see that j_E is involutive. Since the map

$$E \longrightarrow F', \quad x \longmapsto \langle x, \cdot \rangle \mid F$$

is an isometry of Banach spaces, it is an isometry of involutive E-modules. ■

Corollary 4.4.2.11 *Take* $A \subset E$ *and* $B \subset F$. *Put*

$${}^cA := \{a \in F | x \in A \Longrightarrow ax = xa\},$$

$${}^cB := \{x \in E | a \in B \Longrightarrow ax = xa\}.$$

Then ${}^cA$ *is a weakly closed set of* F *and* ${}^cB$ *is a closed set of* E_F. *If* B *is involutive, then* ${}^cB$ *is a closed* C^**-subalgebra of* E_F.

Given $a \in F$ and $x \in E$, define

$$u_a : E_F \longrightarrow F_E, \quad y \longmapsto ay - ya,$$

and

$$v_x : F_E \longrightarrow F_E, \quad b \longmapsto bx - xb.$$

Since

$${}^cA = \bigcap_{x \in A} \operatorname{Ker} v_x, \quad {}^cB = \bigcap_{a \in B} \operatorname{Ker} u_a,$$

the first assertion follows from Corollary 4.4.2.10. The second assertion follows from the first one and Proposition 2.3.6.3. ■

Corollary 4.4.2.12 (0) *If we endow $E^\#$ with the topology of uniform convergence on the weakly compact convex sets of F, then, given $a \in F$, the map*

$$E^\# \longrightarrow \mathbb{K}, \quad x \longmapsto \langle x^* x, a \rangle$$

is continuous.

Take $x_0 \in E^\#$ and $\varepsilon > 0$. Since $E^\#$ is a compact convex set of E_F, the set

$$A := \{ax^* \mid x \in E^\#\}$$

is a weakly compact convex set of F (Corollary 4.4.2.10). Hence there is a neighbourhood V of x_0 in $E^\#$ with

$$|\langle x, b\rangle - \langle x_0, b\rangle| < \frac{\varepsilon}{2}, \quad |\langle x, a^* x_0^*\rangle - \langle x_0, a^* x_0^*\rangle| < \frac{\varepsilon}{2},$$

for every $b \in A$ and $x \in V$. Hence

$$|\langle x^* x, a\rangle - \langle x_0^* x_0, a\rangle| = |\langle x, ax^*\rangle - \langle x_0, ax_0^*\rangle| \leq$$

$$\leq |\langle x, ax^*\rangle - \langle x_0, ax^*\rangle| + |\langle x_0, ax^*\rangle - \langle x_0, ax_0^*\rangle| =$$

$$= |\langle x, ax^*\rangle - \langle x_0, ax^*\rangle| + |\langle x^*, x_0 a\rangle - \langle x_0^*, x_0 a\rangle| =$$

$$= |\langle x, ax^*\rangle - \langle x_0, ax^*\rangle| + |\langle x, a^* x_0^*\rangle - \langle x_0, a^* x_0^*\rangle| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $x \in V$. Thus the map

$$E^\# \longrightarrow \mathbb{K}, \quad x \longmapsto \langle x^* x, a\rangle$$

is continuous. ■

Corollary 4.4.2.13 (0) *Take $x' \in E'_+$. If $x' \leq a$ for some $a \in F$, then x' belongs to F.*

Take $E^{\#}$ with the topology of uniform convergence on the weakly compact convex sets of F. Then

$$|x'(x)|^2 \leq x'(x^*x)x'(1) \leq \langle x^*x, a\rangle x'(1)$$

for every $x \in E^{\#}$ (Proposition 2.3.4.6 c)), so

$$\lim_{x\to 0} x'(x) = 0$$

by Corollary 4.4.2.12. By Proposition 1.3.7.9 a $\Rightarrow$ d, $x' \in \operatorname{Im} j_F$. ■

Corollary 4.4.2.14 *Given $a \in F$, define*

$$\widetilde{a} : E \longrightarrow F, \quad x \longmapsto ax \quad (\textit{resp. } xa)$$

(Corollary 4.4.2.10).

a) *If A is a nonempty downward directed set of F with infimum 0 and if $\mathfrak{F}$ is its lower setion filter, then*

$$\lim_{a,\mathfrak{F}} \widetilde{a} = 0.$$

b) *For every $x \in E$, then map*

$$F \longrightarrow F, \quad a \longmapsto ax \quad (\textit{resp. } xa)$$

is order continuous.

a) First assume $\mathbb{K} = \mathbb{C}$. By Proposition 4.4.1.3, $E_+^{\#}$ is a compact set of E_F and by Theorem 4.4.1.8 a),

$$\lim_{a,\mathfrak{F}} \langle x, a\rangle = 0$$

for every $x \in E$. Choose $\varepsilon > 0$ and $a_0 \in A$. By the above considerations and by Dini's Theorem, there is a $b \in A$, $b \leq a_0$, such that

$$\langle x, a\rangle < \frac{\varepsilon^2}{1 + 16\|a_0\|}$$

for every $a \in A$, $a \leq b$, and $x \in E_+^{\#}$. Thus

$$\|ax\|^2 \leq \langle x, a\rangle \|a\| \, \|x\| \leq \frac{\varepsilon^2}{1+16\|a_0\|}\|a_0\| < \frac{\varepsilon^2}{16}, \quad \|ax\| < \frac{\varepsilon}{4},$$

for every $a \in A$, $a \leq b$, and $x \in E_+^{\#}$, by Proposition 4.3.1.4 (and Corollary 4.2.8.3). It follows that

$$\|ax\| = \|ax^+ - ax^-\| \leq \|ax^+\| + \|ax^-\| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

for every $a \in A$, $a \leq b$, and $x \in \operatorname{Re} E^{\#}$ (Theorem 4.2.2.9 a),b)). Furthermore,

$$\|x^*a\| = \|ax\| = \|a \operatorname{re} x + ia \operatorname{im} x\| \leq \|a \operatorname{re} x\| + \|a \operatorname{im} x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $a \in A$, $a \leq b$, and $x \in E^{\#}$ (Proposition 2.3.2.7). Hence

$$\lim_{a,\mathfrak{F}} \widetilde{a} = 0$$

Let now $\mathbb{K} = \mathbb{R}$. By Theorem 4.4.1.2, the complexification of E is a W^*-algebra and the assertion follows from the above considerations and Corollary 4.3.6.2 d) (and Corollary 4.4.2.13).

b) follows from a). ■

Proposition 4.4.2.15 (0) *Let $\mathfrak{I}$ be the set of closed left ideals of E_F ordered by inclusion. Then $Ep \in \mathfrak{I}$ for every $p \in \operatorname{Pr} E$ and the map*

$$\operatorname{Pr} E \longrightarrow \mathfrak{I}, \quad p \longmapsto Ep$$

is an isomorphism of ordered sets.

Take $p \in \operatorname{Pr} E$. By Proposition 4.4.2.3, the map

$$u : E_F \longrightarrow E_F, \quad x \longmapsto x(1-p)$$

is continuous. From

$$Ep = \operatorname{Ker} u$$

it follows that Ep is closed in E_F. Since it is obviously a left ideal, it belongs to $\mathfrak{I}$.

Take $G \in \mathfrak{I}$. Consider

$$H := G \cap G^*.$$

G^* is obviously a right ideal of E. By Proposition 4.4.1.3, G^* is a closed set of E_F and so H is closed in E_F and a C^*-subalgebra of E. Define

$$A := \{x \in H \mid 0 \leq x \leq 1\}.$$

Let B be a totally ordered nonempty set of A. By Theorem 4.4.1.8 d), the supremum of B in E belongs to A. Hence A is inductively ordered and therefore possesses a maximal element p (Zorn's Lemma). By Proposition 4.2.7.8, $p \in \Pr E$ and $xp = x$ for every $x \in A$.

Take $x \in G^{\#}$. Then $x^*x \in A$ (Corollary 4.2.1.17 a $\Rightarrow$ b) and we get successively that

$$\|x(1-p)\|^2 = \|(1-p)x^*x(1-p)\| = 0,$$

$$x(1-p) = 0,$$

$$x = xp \in Ep,$$

$$G^{\#} \subset Ep,$$

$$G \subset Ep,$$

$$G = Ep.$$

The map

$$\Pr E \longrightarrow \mathfrak{I}, \quad p \longmapsto Ep$$

is therefore surjective.

Take $p, q \in \Pr E$. If $p \leq q$, then

$$p = pq$$

(Corollary 4.2.7.6 a $\Rightarrow$ c) and so

$$Ep \subset Eq.$$

Conversely, if

$$Ep \subset Eq,$$

then

$$p \in Eq,$$

$$pq = p\,,$$

$$p \leq q$$

(Corollary 4.2.7.6 c $\Rightarrow$ a). In particular,

$$Ep = Eq \Longrightarrow p = q$$

and the map

$$\Pr E \longrightarrow \mathfrak{I}\,, \quad p \longmapsto Ep$$

is an isomorphism of ordered sets. ■

Corollary 4.4.2.16 (0) *Let G be a closed ideal of E_F and $u : E \longrightarrow E/G$ the quotient map. Then there is a $p \in \Pr E$ with*

$$G = pE = Ep\,,$$

$(1-p)E(1-p)$ is a W^*-algebra with $1-p$ as unit, and the map

$$(1-p)E(1-p) \longrightarrow E/G\,, \quad x \longmapsto ux$$

is an isometry of unital C^*-algebras (Theorem 4.2.6.5).

By Proposition 4.4.2.15, there are $p, q \in \Pr E$ with

$$G = pE = Eq\,.$$

We deduce that

$$p \in Eq\,, \quad q \in pE\,,$$

$$p = pq = q\,.$$

Put

$$v : (1-p)E(1-p) \longrightarrow E/G\,, \quad x \longmapsto ux\,.$$

Then

$$x = px = 0$$

for every $x \in \operatorname{Ker} v$. Thus v is injective. Let $y \in E$. We have

$$y - (1-p)y(1-p) = py + yp - pyp\,,$$

so that

$$v((1-p)y(1-p)) = uy\,.$$

Hence v is surjective and

$$v(1-p) = u1\,.$$

v is therefore an isometry of unital C^*-algebras (Theorem 4.2.6.6). By Proposition 4.4.2.2, $(1-p)E(1-p)$ is a W^*-algebra. ■

Corollary 4.4.2.17 (0) *Every element of F has a right and a left carrier. Every element of* $\operatorname{Re} F$ *has a carrier.*

Take $a \in F$. Consider

$$G := \{x \in E \mid ax = 0\} \quad (G := \{x \in E \mid xa = 0\})\,.$$

G is a closed right (left) ideal of E_F (Corollary 4.4.2.10). By Proposition 4.4.2.15, there is a $q \in \operatorname{Pr} E$ with

$$G = qE \quad (G = Eq)\,.$$

Put

$$p := 1 - q\,.$$

Then

$$aq = 0 \quad (qa = 0)\,,$$

so that

$$ap = a \quad (pa = a)\,.$$

Since

$$pG = pqE = \{0\} \quad (Gp = Eqp = \{0\})\,,$$

p is the right (left) carrier of a.

The last assertion follows from the first one and from Proposition 4.3.3.3. ■

Corollary 4.4.2.18 *Take $a, b \in F_+$, $a \leq b$. If p and q are the carriers of a and b, respectively (Corollary 4.4.2.17), then $p \leq q$.*

We see successively that

$$b(1-q) = 0\,,$$

$$0 \leq \langle 1-q, a\rangle \leq \langle 1-q, b\rangle = \langle 1, b(1-q)\rangle = 0\,,$$

$$\langle 1-q, a\rangle = 0\,,$$

$$a(1-q) = 0$$

(Corollary 4.3.1.5 a $\Rightarrow$ c),

$$a = aq\,,$$

$$p \leq q$$

(Proposition 4.3.3.2 b)). ■

Proposition 4.4.2.19 *Let $\mathfrak{C}$ be the set of faces of E_+ which are closed in E_F. Then $pE_+p \in \mathfrak{C}$ for each $p \in \Pr E$ and the map*

$$\Pr E \longrightarrow \mathfrak{C}\,, \quad p \longmapsto pE_+p$$

is bijective.

By Corollary 4.3.4.4, pE_+p is a face of E_+ whenever $p \in \Pr E$ and by Propositions 4.4.1.3 and 4.4.2.2, this face is closed in E_F. It is obvious that the map

$$\Pr E \longrightarrow \mathfrak{C}\,, \quad p \longmapsto pE_+p$$

is injective.

In order to show that this map is surjective, take $C \in \mathfrak{C}$. Choose $p, q \in C \cap \Pr E$. By Theorem 4.4.1.8 i), $p \vee q \in \Pr E$ and

$$p \vee q \leq p + q\,.$$

Since $p + q \in C$, it follows from Proposition 4.3.4.3 that $p \vee q \in C$. Hence the set $C \cap \Pr E$ is upward directed. Let $\mathfrak{F}$ be its upper section filter. By

Theorem 4.4.1.8 b),i), $\mathfrak{F}$ converges in E_F to the supremum p of $C \cap \Pr E$ and $p \in C \cap \Pr E$.

Take $x \in C$. By Corollary 4.4.1.13, the carrier q of x belongs to C. Thus

$$q \leq p\,,$$

$$xp = xqp = xq = x$$

(Corollary 4.2.7.6 a $\Rightarrow$ c). It follows that

$$x = pxp \in pE_+p\,,$$

$$C \subset pE_+p\,.$$

Take $y \in pE_+p$. Then

$$y \leq \|y\|p\,,$$

$$y \in C\,,$$

$$pE_+p \subset C\,,$$

$$pE_+p = C\,.$$

Hence the map

$$\Pr E \longrightarrow \mathfrak{C}\,, \quad p \longmapsto pE_+p$$

is surjective. ■

Theorem 4.4.2.20 (0) *Let G be a closed C^*-subalgebra of E_F. If $u : G \to E$ is the inclusion map, then $u'(F)$ is an involutive G-submodule of G' and the map*

$$G \longrightarrow (u'(F))'\,, \quad x \longmapsto \langle x, \cdot \rangle | u'(F)$$

is an isometry.

By Theorem 1.3.5.12 c), the factorization $E'/G^\circ \to G'$ of u' through E'/G° is an isometry. Its restriction

$$v : F/(F \cap G^\circ) \longrightarrow u'(F)$$

is also an isometry. By Proposition 2.2.7.7 (and Corollary 4.4.2.10), $u'(F)$ is a G–submodule of G' and it is easy to see, that it is an involutive G–submodule of G'.

Let

$$q : F \longrightarrow F/(F \cap G^\circ)$$

be the quotient map. By Corollary 1.3.5.5, the map

$$\psi : (F/(F \cap G^\circ))' \longrightarrow G\,, \quad x \longmapsto q'x$$

is an isometry.

Put

$$\varphi : G \longrightarrow (u'(F))'\,, \quad x \longmapsto \langle x, \cdot\rangle | u'(F)\,.$$

Take $x \in G$ and $a \in F$. Then

$$\langle \psi((\varphi x) \circ v), a\rangle = \langle q'((\varphi x) \circ v), a\rangle = \langle (\varphi x) \circ v, qa\rangle =$$

$$= \langle \varphi x, vqa\rangle = \langle \varphi x, u'a\rangle = \langle x, u'a\rangle = \langle ux, a\rangle = \langle x, a\rangle\,.$$

Hence

$$\psi((\varphi x) \circ v) = x\,,$$

$$(\varphi x) \circ v = \psi^{-1}x\,,$$

$$\varphi x = (\psi^{-1}x) \circ v^{-1}\,,$$

and the map

$$\varphi : G \longrightarrow (u'(F))'\,, \quad x \longmapsto (\psi^{-1}x) \circ x^{-1}$$

is an isometry. ■

Theorem 4.4.2.21 (0) *A real C^*–algebra is a real W^*–algebra iff its complexification is a complex W^*–algebra.*

Let G be a real C^*-algebra and H a closed vector subspace of $(\overset{\circ}{G})'$ such that the map

$$\overset{\circ}{G} \longrightarrow H', \quad z \longmapsto \langle z, \cdot \rangle | H$$

is an isometry. Put

$$I := \{a \in G' \mid \exists\, b \in G',\ (a,b) \in H\}.$$

I is obviously an involutive vector subspace of G' (Corollary 4.4.2.10). Take $x \in G$ and $\alpha \in [0, \|x\|[$. Then

$$\alpha < \|x\| = \|(x,0)\| = \|\langle (x,0), \cdot \rangle | H\|.$$

Hence there is an $(a,b) \in H^{\#}$ such that

$$\alpha < |\langle (x,0), (a,b) \rangle| = |\langle x, a \rangle + i \langle x, b \rangle|.$$

Then

$$(\langle x,a\rangle a + \langle x,b\rangle b, -\langle x,b\rangle a + \langle x,a\rangle b) = (\langle x,a\rangle - i\langle x,b\rangle))(a,b) \in H,$$

$$\|\langle x,a\rangle a + \langle x,b\rangle b\| \leq \|(\langle x,a\rangle a + \langle x,b\rangle b, -\langle x,b\rangle a + \langle x,a\rangle b)\| =$$

$$= (\langle x,a\rangle^2 + \langle x,b\rangle^2)^{\frac{1}{2}} \|(a,b)\| \leq (\langle x,a\rangle^2 + \langle x,b\rangle^2)^{\frac{1}{2}}$$

(Corollary 4.3.6.2 a)). Thus

$$\|\langle x,\cdot\rangle | I\| (\langle x,a\rangle^2 + \langle x,b\rangle^2)^{\frac{1}{2}} \geq$$

$$\geq |\langle x, \langle x,a\rangle a + \langle x,b\rangle b\rangle| = \langle x,a\rangle^2 + \langle x,b\rangle^2,$$

$$\|\langle x,\cdot\rangle | I\| \geq (\langle x,a\rangle^2 + \langle x,b\rangle^2)^{\frac{1}{2}} > \alpha.$$

Since α is arbitrary,

$$\|x\| \leq \|\langle x,\cdot\rangle | I\| \leq \|\langle x,\cdot\rangle\| = \|x\|,$$

$$\|\langle x,\cdot\rangle | I\| = \|x\|.$$

Hence the map

$$G \longrightarrow I', \quad x \longmapsto \langle x, \cdot \rangle | I$$

is an isometry.

Let $(x, a) \in G \times I$. There is a $b \in G'$ such that $(a, b) \in H$. Then

$$(xa, xb) = (x, 0)(a, b) \in H,$$

$$(ax, bx) = (a, b)(x, 0) \in H$$

(Corollary 4.4.2.10), so that $xa, ax \in I$. The closure of I in G' is an involutive E submodule of G', so that E is a real W^*-algebra.

The converse implication was proved in Theorem 4.4.1.2. ■

Proposition 4.4.2.22 *Suppose* $\mathbb{K} = \mathbb{C}$ *and let* $\widetilde{E}$ *denote the underlying real* C^**-algebra of* E *(Theorem 4.1.1.8 a)). Put*

$$\widetilde{x'} : E \longrightarrow \mathbb{R}, \quad x \longmapsto \operatorname{re} \langle x, x' \rangle$$

for every $x' \in E'$ *and*

$$\widetilde{F} := \{\widetilde{a} | a \in F\}.$$

a) $\widetilde{x'} \in (\widetilde{F})'$ *for every* $x' \in E'$ *and the map*

$$E' \longrightarrow (\widetilde{E})', \quad x' \longmapsto \widetilde{x'}$$

is an isometry of involutive $\widetilde{E}$*-modules.*

b) $\widetilde{F}$ *is an involutive* $\widetilde{E}$*-submodule of* $(\widetilde{E})'$ *and the map*

$$\widetilde{E} \longrightarrow (\widetilde{F})', \quad x \longmapsto \langle \cdot, x \rangle | \widetilde{F}$$

is an isometry of involutive $\widetilde{E}$*-modules.*

c) $\widetilde{E}$ *is a real* W^**-algebra with* $\widetilde{F}$ *as predual.*

d) *The complexification of* E *is a complex* W^**-algebra.*

a) follows from Proposition 2.3.6.8.
b) follows from a) and Proposition 1.3.6.26 c).
c) follows from b).
d) follows from c) and Theorem 4.4.2.21 (and Theorem 4.1.1.8 a)). ■

Proposition 4.4.2.23 (0) *For every $x \in E_+$, the space*

$$\{y \in E \mid y^*y \leq x\}_F$$

is compact.

By Theorem 4.4.2.4, the map

$$\varphi : E_F \longrightarrow E_F\,, \quad y \longmapsto yx^{\frac{1}{2}}$$

is continuous so that by the Theorem of Alaoglu–Bourbaki, $\varphi(E^\#)_F$ is compact. We want to prove

$$\{y \in E \mid y^*y \leq x\} = \varphi(E^\#)$$

which proves the proposition.

Take $y \in E^\#$. Then $y^*y \leq 1$ (Corollary 4.2.1.17 a $\Rightarrow$ b), so that

$$(\varphi y)^*(\varphi y) = (yx^{\frac{1}{2}})^*(yx^{\frac{1}{2}}) = x^{\frac{1}{2}}y^*yx^{\frac{1}{2}} \leq x^{\frac{1}{2}}x^{\frac{1}{2}} = x$$

(Corollary 4.2.2.3). Thus

$$\varphi(E^\#) \subset \{y \in E \mid y^*y \leq x\}\,.$$

Now take $y \in E$ with $y^*y \leq x$. For every $n \in \mathbb{N}$ put

$$A_n := \{\alpha \in \sigma(x) \mid \alpha \geq \frac{1}{n}\}\,,$$

$$f_n : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \begin{cases} \alpha^{-\frac{1}{2}} & \text{if } \alpha \in A_n \\ 0 & \text{if } \alpha \in \sigma(x)\backslash A_n\,, \end{cases}$$

$$x_n := f_n(x)\,, \quad y_n := yx_n\,.$$

Then for $n \in \mathbb{N}$,

$$y_n^*y_n = x_ny^*yx_n \leq x_nxx_n = e_{A_n}^{\sigma(x)}(x) \leq 1$$

(Corollary 4.2.2.3, Corollary 4.3.2.5 a),b)) and $y_n \in E^\#$. Hence

$$ye_{A_n}^{\sigma(x)}(x) = yx_nx^{\frac{1}{2}} = y_nx^{\frac{1}{2}} \in \varphi(E^\#)\,.$$

Put

$$p := e^{\sigma(x)}_{\sigma(x)\setminus\{0\}}(x)\,.$$

By Theorem 4.3.3.8 a $\Rightarrow$ b (and Theorem 4.4.1.8 j), Corollary 4.3.3.9), p is the carrier of x. By Proposition 4.3.3.2 d),

$$y^*y = y^*yp\,,$$

so that

$$\|y - yp\|^2 = \|(y^* - py^*)(y - yp)\| = \|y^*y - py^*y - y^*yp + py^*yp\| = 0\,,$$

$$y = yp\,.$$

Since $\left(e^{\sigma(x)}_{A_n}(x)\right)_{n\in\mathbb{N}}$ is an increasing sequence in E with p as supremum (Theorem 4.3.2.5 a),b)), it follows that it converges to p in E_F (Theorem 4.4.1.8 b)). Thus for every $a \in F$,

$$\langle y, a\rangle = \langle yp, a\rangle = \langle p, ay\rangle = \lim_{n\to\infty}\left\langle e^{\sigma(x)}_{A_n}(x), ay\right\rangle =$$

$$= \lim_{n\to\infty}\left\langle ye^{\sigma(x)}_{A_n}(x), a\right\rangle\,.$$

Hence $\left(ye^{\sigma(x)}_{A_n}(x)\right)_{n\in\mathbb{N}}$ converges to y in E_F and $y \in \varphi(E^\#)$,

$$\{y \in E \mid y^*y \leq x\} \subset \varphi(E^\#)\,. \qquad \blacksquare$$

Proposition 4.4.2.24 (Kaplansky) *Let* $p, q \in \Pr E$.

a) $(1-p)\wedge(1-q) = 1 - p\vee q$.

b) $p - p\wedge q$ *is the right carrier of* $(1-q)p$.

c) $p\vee q - q$ *is the left carrier of* $(1-q)p$.

a) From

$$1 - p\vee q \leq 1 - p\,, \quad 1 - p\vee q \leq 1 - q\,,$$

it follows

$$1 - p\vee q \leq (1-p)\wedge(1-q)\,.$$

From

$$1-(1-p)\wedge(1-q)\geq 1-(1-p)=p\,,$$

$$1-(1-p)\wedge(1-q)\geq 1-(1-q)=q$$

it follows

$$p\vee q\leq 1-(1-p)\wedge(1-q)\,,$$

$$(1-p)\wedge(1-q)\leq 1-p\vee q\,,$$

$$(1-p)\wedge(1-q)=1-p\vee q\,.$$

b) By Corollary 4.2.7.6 a $\Rightarrow$ b,

$$(1-q)p(p-p\wedge q)=(1-q)(p-p\wedge q)=$$

$$=p-p\wedge q-qp+p\wedge q=p-qp=(1-q)p\,.$$

Take $x\in E$ with

$$(1-q)px=0\,.$$

Then

$$px=qpx\,,$$

so

$$px=pqpx\,.$$

By induction

$$px=(pqp)^n x$$

for every $n\in\mathbb{N}$. $((pqp)^n)_{n\in\mathbb{N}}$ is a decreasing sequence in E_+ with infimum $p\wedge q$ (Corollary 4.2.7.13). By Theorem 4.4.1.8 c),

$$\lim_{n\to\infty}\langle(pqp)^n,a\rangle=\langle p\wedge q,a\rangle\,,$$

so that

$$\langle(p\wedge q)x,a\rangle=\langle p\wedge q,xa\rangle=\lim_{n\to\infty}\langle(pqp)^n,xa\rangle=\lim_{n\to\infty}\langle(pqp)^n x,a\rangle$$

for every $a \in F$ (Corollary 4.4.2.10). It follows

$$\langle (p - p \wedge q)x, a\rangle = \langle px, a\rangle - \langle (p \wedge q)x, a\rangle =$$

$$= \langle px, a\rangle - \lim_{n\to\infty} \langle (pqp)^n x, a\rangle = \langle px, a\rangle - \langle px, a\rangle = 0$$

for every $a \in F$ and therefore

$$(p - p \wedge q)x = 0\,.$$

Hence $p - p \wedge q$ is the right carrier of $(1 - q)p$.

c) Put

$$p^\perp := 1 - p\,, \quad q^\perp := 1 - q\,.$$

By b), $p^\perp - p^\perp \wedge q^\perp$ is the right carrier of

$$(1 - q^\perp)p^\perp = q(1 - p) = ((1 - p)q)^*\,.$$

Hence $p^\perp - p^\perp \wedge q^\perp$ is the left carrier of $(1 - p)q$ (Proposition 4.3.3.3). By a),

$$p^\perp - p^\perp \wedge q^\perp = 1 - p - (1 - p \vee q) = p \vee q - p\,.$$

■

Remark. a) follows also from Corollary 4.2.7.13 and Proposition 4.2.7.14.

4.4.3 Polar Representation

Theorem 4.4.3.1 (0) *For every $x \in E$, there is a unique $y \in E$ with*

1) $x = y|x|$;

2) y^*y *is the carrier of* $|x|$ *and the right carrier of* x;

3) yy^* *is the carrier of* $|x^*|$ *and the left carrier of* x.

$(y, |x|)$ is called ***the polar representation of*** *x. If x is invertible then the above definition coincides with the one given in Theorem 4.2.6.9.*

Given $n \in \mathbb{N}$, put

$$x_n := \left(\frac{1}{n}1 + x^*x\right)^{\frac{1}{2}},$$

$$y_n := x\left(\frac{1}{n}1 + x^*x\right)^{-\frac{1}{2}}.$$

Then

$$y_n^*y_n = \left(\frac{1}{n}1 + x^*x\right)^{-\frac{1}{2}} x^*x \left(\frac{1}{n}1 + x^*x\right)^{-\frac{1}{2}} = x^*x\left(\frac{1}{n}1 + x^*x\right)^{-1},$$

so that

$$\|y_n\| \leq 1$$

for every $n \in \mathbb{N}$. Let z be a point of adherence of $(y_n)_{n\in\mathbb{N}}$ in $E_F^{\#}$ (Alaoglu–Bourbaki Theorem). Then $z|x|$ is a point of adherence of $(y_n|x|)_{n\in\mathbb{N}}$ in E_F (Theorem 4.4.2.4).

Take $\varepsilon > 0$. Since

$$\lim_{n\to\infty} x_n = |x|,$$

there is an $n_0 \in \mathbb{N}$ with

$$\|x_n - |x|\| < \varepsilon$$

for every $n \in \mathbb{N}$, $n \geq n_0$. Thus

$$\|y_n|x| - x\| = \|y_n|x| - y_nx_n\| = \|y_n(|x| - x_n)\| \leq \|\,|x| - x_n\| < \varepsilon$$

for every $n \in \mathbb{N}$, $n \geq n_0$ and hence

$$\|z|x| - x\| \leq \limsup_{n\to\infty} \|y_n|x| - x\| \leq \varepsilon$$

(Proposition 1.2.6.6). Since ε is arbitrary, we deduce that

$$x = z|x|\,.$$

Let p (resp. q) be the carrier of $|x|$ (of $|x^*|$) (Theorem 4.4.1.8 j)) and put

$$y := qzp\,.$$

By Corollary 4.3.3.10 b $\Rightarrow$ a & c, p (resp. q) is the right (left) carrier of x. Then

$$x = qx = qz|x| = qzp|x| = y|x|\,,$$

$$|x|p|x| = |x|^2 = x^*x = |x|y^*y|x|\,,$$

$$|x|(p - y^*y)|x| = 0\,.$$

Since $z \in E^{\#}$, we get successively that

$$\|z^*z\| = \|z\|^2 \leq 1\,,$$

$$z^*z \leq 1$$

(Corollary 4.2.1.17 a $\Rightarrow$ b),

$$y^*y = pz^*qzp \leq pz^*zp \leq p$$

(Corollary 4.2.2.3),

$$0 \leq p - y^*y \leq p\,,$$

$$(p - y^*y)^{\frac{1}{2}} \leq p$$

(Corollary 4.2.7.2),

$$\|(p - y^*y)^{\frac{1}{2}}|x|\,\|^2 = \|\,|x|(p - y^*y)|x|\,\| = 0\,,$$

$$(p - y^*y)^{\frac{1}{2}}|x| = 0\,,$$

$$(p - y^*y)^{\frac{1}{2}} = (p - y^*y)^{\frac{1}{2}}p = 0$$

(Proposition 4.2.7.1 d $\Rightarrow$ f),

$$y^*y = p\,.$$

We have that

$$yy^* = qzpz^*q \leq qzz^*q \leq q$$

(Corollary 4.2.2.3). By Corollary 4.1.2.22, $yy^* \in \Pr E$ and

$$xx^*yy^* = x|x|y^*yy^* = x|x|y^* = xx^*\,,$$

so that

$$|x^*|yy^* = |x^*|$$

(Proposition 4.2.7.1 d $\Leftrightarrow$ f, Corollary 4.2.7.2)

$$q \leq yy^*$$

(Proposition 4.3.3.2 b)),

$$yy^* = q\,.$$

We now prove the uniqueness of y. Take $z \in E$, so that

$$x = z|x|\,, \quad z^*z = p\,, \quad zz^* = q\,.$$

We get successively that

$$z^*y|x| = z^*x = z^*z|x| = p|x|\,,$$

$$(p - z^*y)|x| = 0\,,$$

$$(p - z^*y)p = 0\,,$$

$$p - z^*y = p - z^*yy^*y = p - z^*yp = 0$$

(Corollary 4.1.2.22),

$$z = zz^*z = zp = zz^*y = qy = yy^*y = y$$

(Corollary 4.1.2.22).

If x is invertible then the left and the right carrier of x are 1 (Proposition 4.3.3.5 b)). Thus y is unitary and the above polar representation of x coincides with the one defined in Theorem 4.2.6.9. ■

Corollary 4.4.3.2 (0) *Take $x \in E$ and let $(y, |x|)$ be its polar representation.*

a) *The following are equivalent:*

a_1) *x is normal.*

a_2) *y is normal and*

$$y|x| = |x|y\,.$$

a_3) *y is normal and*

$$xy = yx\,.$$

If x is self-normal then $y = f(x)$, where

$$f : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \begin{cases} \frac{\alpha}{|\alpha|} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0\,. \end{cases}$$

b) *The following are equivalent:*

b_1) *x is selfadjoint.*

b_2) *y is selfadjoint and*

$$y|x| = |x|y\,.$$

b_3) *y is selfadjoint and*

$$xy = yx\,.$$

If these conditions are fulfilled then

$$x^+ = y^+|x| = y^+x^+\,, \quad x^- = y^-|x| = y^-x^-\,.$$

$a_1 \Rightarrow a_2$. We have

$$|x| = |x^*|\,,$$

so that

$$y^*y = yy^*\,,$$

i.e. y is normal. We deduce successively that

$$|x|^2 = xx^* = y|x|^2y^*,$$

$$|x|^2y = y|x|^2y^*y = y|x|^2,$$

$$|x|y = y|x|$$

(Proposition 4.2.4.2 b)).

$a_2 \Rightarrow a_1$. We have

$$xx^* = y|x|^2y^* = |x|^2yy^* = |x|^2y^*y = |x|^2 = x^*x.$$

$a_2 \Rightarrow a_3$. By Theorem 4.1.4.1,

$$y^*x = y^*y|x| = |x|y^*y = |x|,$$

$$xy^* = |x|yy^* = |x|y^*y = |x| = y^*x,$$

$$xy = yx.$$

$a_3 \Rightarrow a_2$. By Theorem 4.1.4.1,

$$xy^* = y^*x,$$

so that

$$yx^* = x^*y,$$

$$y|x|^2 = yx^*x = x^*xy = |x|^2y,$$

and

$$y|x| = |x|y$$

by Proposition 4.2.4.2 b).

Now we prove the final assertion in a). Put

$$j : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha,$$

$$g : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto |\alpha|.$$

Then

$$g(x) = |x|$$

(Proposition 4.2.5.4) and

$$fg = j$$

so that

$$f(x)|x| = x\,.$$

From

$$|f|^2 = e^{\sigma(x)}_{\sigma(x)\setminus\{0\}}$$

it follows

$$f^*(x)f(x) = |f|^2(x) = e^{\mathbb{K}}_{\mathbb{K}\setminus\{0\}}(x) = e^{\mathbb{K}}_{\mathbb{K}\setminus\{0\}}(|x|)\,.$$

By Corollary 4.3.3.14, $f^*(x)f(x)$ is the carrier of x and of $|x|$ so that $(f(x), |x|)$ is the polar representation of x. Hence

$$y = f(x)\,.$$

b) First suppose that x is selfadjoint. By a), y is normal and

$$y|x| = |x|y\,, \quad xy = yx\,.$$

We get

$$x = x^* = y^*|x|\,,$$

and so $(y^*, |x|)$ is the polar representation of x. This implies that $y^* = y$. Moreover,

$$y^+|x| - y^-|x| = y|x| = x\,,$$

and by the Fuglede–Putnam Theorem,

$$(y^+|x|)(y^-|x|) = y^+y^-|x|^2 = 0\,,$$

$$(y^-|x|)(y^+|x|) = y^-y^+|x|^2 = 0\,.$$

Hence

$$x^+ = y^+|x|\,, \quad x^- = y^-|x|\,.$$

It follows that

$$y^+x^- = y^+y^-|x| = 0\,, \quad y^-x^+ = y^-y^+|x| = 0\,,$$

so that

$$y^+|x| = y^+(x^+ + x^-) = y^+x^+\,, \quad y^-|x| = y^-(x^+ + x^-) = y^-x^-\,.$$

The implication $b_2 \Rightarrow b_1$ is trivial. The implication $b_3 \Rightarrow b_2$ follows from a) and the Fuglede–Putnam Theorem. ■

Proposition 4.4.3.3 (0) *Take $a \in F$. There is an $x \in E$ such that $x^*x \in \Pr E$, ax, $xa \in F_+$,*

$$x^*xa = axx^* = a\,.$$

If a is selfadjoint, then we may take x to be also selfadjoint.

We may assume that $\|a\| = 1$. There is a $y \in E^\#$ with

$$\langle a, y\rangle = 1\,.$$

By Theorem 1.3.1.10 b), there is an extreme point x of $E^\#$ with

$$\langle a, x\rangle = 1\,.$$

By Theorem 4.3.3.20 a $\Rightarrow$ b & d, x^*x, $xx^* \in \Pr E$ and

$$(1 - xx^*)E(1 - x^*x) = \{0\}\,.$$

Thus

$$(1 - x^*x)a(1 - xx^*) = 0\,,$$

$$a(1 - xx^*) = x^*xa(1 - xx^*)\,.$$

Now

$$1 = \langle x, a\rangle = \langle 1, ax\rangle \leq \|ax\| \leq \|a\|\,\|x\| \leq 1\,,$$

$$1 = \langle x, a\rangle = \langle 1, xa\rangle \leq \|xa\| \leq \|x\|\,\|a\| \leq 1\,,$$

$$\langle 1, ax\rangle = \|ax\|\,, \quad \langle 1, xa\rangle = \|xa\|$$

and $ax, xa \in F_+$ (Corollary 4.3.6.1 g)). We deduce that

$$ax = (ax)^* = x^*a^*, \quad xa = (xa)^* = a^*x^*,$$

$$axx^* = x^*a^*x^* = x^*xa,$$

$$a(1 - xx^*) = axx^*(1 - xx^*) = 0,$$

$$x^*xa = axx^* = a.$$

If a is selfadjoint, then we may take the above y to be selfadjoint (Proposition 2.3.2.22 l)) and x to be an extreme point of $\operatorname{Re} E^\#$. By Proposition 4.3.3.21 a $\Rightarrow$ c, x is an extreme point of $E^\#$, so the above considerations apply. ■

Proposition 4.4.3.4 (0) *Take* $(y, a) \in E \times F$ *with* $y^*y \in \Pr E$, $ya \in F_+$, *and*

$$y^*ya = a.$$

Put

$$b := ya.$$

Let q *be the carrier of* b *(Corollary 4.4.2.17) and put*

$$x := y^*q.$$

Then

$$x^*x = q, \quad xb = a, \quad b = x^*a, \quad \|a\| = \|b\|.$$

We have $yy^* \in \Pr E$ (Corollary 4.1.2.22 a $\Rightarrow$ b,) and

$$y^*b = y^*ya = a,$$

$$yy^*b = ya = b,$$

so that

$$q \leq yy^*$$

(Proposition 4.3.3.2 b)),

$$x^*x = qyy^*q = q$$

(Corollary 4.2.7.6 a $\Rightarrow$ e),

$$xb = y^*qb = y^*b = y^*ya = a\,,$$

$$b = qb = x^*xb = x^*a\,.$$

We deduce that

$$\|a\| = \|xb\| \leq \|x\|\,\|b\| \leq \|b\| = \|x^*a\| \leq \|x^*\|\,\|a\| \leq \|a\|\,,$$

$$\|a\| = \|b\|\,. \qquad \blacksquare$$

Theorem 4.4.3.5 (0) (Sakai, 1958) *Let* $a \in F$.

a) *There is a unique* $(x, b) \in E \times F_+$ *such that* x^*x *is the carrier of* b *and*

$$a = xb\,.$$

We set

$$|a| := b \qquad (\textbf{\textit{the modulus of}}\, a)\,,$$

and call (x, b) ***the polar representation of*** a.

b) xx^* *is the carrier of* $|a^*|$ *and let left carrier of* a.

c) *If* $s \in E^c$ *and if* $(t, |s|)$ *is the polar representation of* s *(Theorem 4.4.3.1), then* $(tx, |s|\,|a|)$ *is the polar representation of* sa. *In particular,* $|sa| = |s|\,|a|$.

Step 1 Uniqueness in a)

By Theorem 4.4.1.2 (or Corollary 4.4.3.7), we may assume $\mathbb{K} = \mathbb{C}$. Take $(x, b), (y, c) \in E \times F_+$ with

$$xb = yc\,,$$

such that x^*x (resp. y^*y) is the carrier of b (of c). We have, successively, that

$$b = x^*xb = x^*yc = x^*ycy^*y\,,$$

$$b = by^*y\,,$$

$$x^*x \leq y^*y$$

(Proposition 4.3.3.2 b)),

$$x^*x = y^*y\,,$$

$$\|xb\| \leq \|x\|\,\|b\| \leq \|b\| \leq \|x^*\|\,\|xb\| \leq \|xb\|\,,$$

$$\|b\| = \|xb\| = \|yc\| = \|c\|\,.$$

Put

$$p := x^*x\,, \quad u := \mathrm{re}\,(x^*y)\,, \quad v := \mathrm{im}\,(x^*y)\,.$$

Then

$$px^* = x^*xx^* = x^*\,, \quad py^* = y^*yy^* = y^*$$

(Corollary 4.1.2.22 a $\Rightarrow$ c), so that

$$pu = \frac{1}{2}p(x^*y + y^*x) = \frac{1}{2}(x^*y + y^*x) = u\,,$$

$$pv = \frac{1}{2i}p(x^*y - y^*x) = \frac{1}{2i}(x^*y - y^*x) = v\,,$$

$$up = (pu)^* = u\,, \quad vp = (pv)^* = v\,.$$

Since $\langle c, u\rangle\,,\ \langle c, v\rangle \in \mathbb{R}$ (Corollary 2.3.1.23 a $\Rightarrow$ b) and

$$\langle c, u\rangle + i\langle c, v\rangle = \langle c, x^*y\rangle = \langle x^*yc, 1\rangle = \langle b, 1\rangle = \|b\| = \|c\| = \langle c, 1\rangle$$

(Corollary 2.3.4.7), we see that

$$\langle c, u\rangle = \|c\|\,,$$

$$\langle c, p - u\rangle = \langle c, p\rangle - \langle c, u\rangle = \langle pc, 1\rangle - \|c\| = \langle c, 1\rangle - \|c\| = 0\,.$$

From

$$\|u\| \leq \|x^*y\| \leq \|x^*\|\,\|y\| \leq 1$$

(Proposition 2.3.2.7), we deduce successively that

$$u \leq 1$$

(Corollary 4.2.1.16 a $\Rightarrow$ b),

$$u = pup \leq p1p = p$$

(Corollary 4.2.2.3),

$$p - u \geq 0\,,$$

$$(p-u)c = 0$$

(Corollary 4.3.1.5 a $\Rightarrow$ b),

$$0 = (p-u)p = p - u\,,$$

$$p = u\,,$$

$$x^*y = u + iv = p + iv\,,$$

$$1 \geq \|x^*y\|^2 = \|p + iv\|^2 = \|(p - iv)(p + iv)\| = \|p + v^2\|\,.$$

The C^*-subalgebra G of E generated by p and v is a Gelfand C^*-algebra (Corollary 4.1.2.3). Identifying G with $\mathcal{C}_0(\sigma(G))$ and using the above relation, we get

$$1 \geq \|p + v^2\| = \|p\| + \|v\|^2 = 1 + \|v\|^2\,,$$

since we may assume $p \neq 0$. Hence

$$\|v\| = 0\,,$$

$$v = 0\,,$$

$$x^*y = u = p\,,$$

$$b = pc = c\,,$$

$$(x-y)b = xb - yc = 0\,,$$

$$x - y = xx^*x - yy^*y = xp - yp = (x-y)p = 0$$

(Corollary 4.1.2.22 a $\Rightarrow$ c),

$$x = y\,.$$

Step 2 The existence in a) and b)

By Proposition 4.4.3.3, there is a $z \in E$ with $z^*z \in \operatorname{Pr} E$, az, $za \in F_+$, and

$$z^*za = azz^* = a\,.$$

Put

$$b := za \in F_+\,, \quad c := z^*a^* = (az)^* = az \in F_+\,,$$

and let p (resp. q) denote the carrier of b (resp. c) (Corollary 4.4.2.17). Put

$$x := z^*p\,, \quad y := zq\,.$$

By Proposition 4.4.3.4 (and Corollary 4.1.2.22)

$$x^*x = p\,, \quad xb = a\,, \quad b = x^*a\,, \quad \|a\| = \|b\|\,,$$

$$y^*y = q\,, \quad yc = a^*\,, \quad c = y^*a^*\,.$$

Thus $xx^* \in \operatorname{Pr} E$,

$$xp = xx^*x = x$$

(Corollary 4.1.2.22),

$$ap = xbp = xb = a\,,$$

$$pa^* = a^*\,,$$

$$(qa)^* = a^*q = ycq = yc = a^*\,,$$

$$qa = a\,.$$

We deduce successively that

$$x^*c = pzc = pzz^*a^* = pa^* = a^*\,,$$

$$xx^*c = xa^* = xbx^* = ax^* = apz = az = c\,,$$

$$y^*y = q \leq xx^*$$

(Proposition 4.3.3.2 b)),

$$(qx - x)b = qxb - xb = qa - a = 0\,,$$

$$0 = (qx - x)p = qxp - xp = qx - x\,,$$

$$qxx^* = xx^*\,,$$

$$xx^* \leq q$$

(Corollary 4.2.7.6 d $\Rightarrow$ a),

$$xx^* = q\,.$$

Let $w \in E$ such that

$$wa = 0\,.$$

Then

$$wx|a| = wa = 0\,,$$

and we get

$$0 = wxp = wxx^*x = wx\,,$$

$$wq = wxx^* = 0$$

(Corollary 4.1.2.22). Hence q is the left carrier of a.

Step 3 c)

E^c is a C^*-subalgebra of E (Corollary 4.1.4.2 b)) and $|s| \in E^c$,

$$|s|\,|a| = |a|\,|s| \in F_+$$

(Proposition 4.2.4.2 d),g), Corollary 4.4.2.10). For every $r \in E$, we have successively,

$$rt^*t|s| = r|s| = |s|r = t^*t|s|r = t^*tr|s|\,,$$

$$(rt^*t - t^*tr)|s| = 0\,,$$

$$0 = (rt^*t - t^*tr)t^*t = rt^*t - t^*trt^*t\,,$$

$$rt^*t = t^*trt^*t\,,$$

$$r^*t^*t = t^*tr^*t^*t\,,$$

$$t^*tr = t^*trt^*t = rt^*t\,,$$

so that $t^*t \in E^c$. It follows

$$(tx)^*(tx) = x^*t^*tx = t^*tx^*x \in \Pr E$$

(Corollary 4.2.7.4 a $\Rightarrow$ b),

$$(tx)^*(tx)|s|\,|a| = t^*tx^*x|a|\,|s| = t^*t|a|\,|s| = t^*t|s|\,|a| = |s|\,|a|\,.$$

Take $r \in E$ with

$$r|s|\,|a| = 0\,.$$

Then

$$0 = (r|s|)x^*x = rx^*x|s|\,,$$

$$0 = rx^*xt^*t = r(tx)^*(tx)\,.$$

Hence $(tx)^*(tx)$ is the carrier of $|s|\,|a|$ and $(tx, |s|\,|a|)$ is the polar representation of sa. ■

Corollary 4.4.3.6 *Take $a \in F$ and let $(x, |a|)$ be its polar representation. Choose $y \in E$ such that*

$$xx^* \leq y^*y \in \Pr E\,.$$

(This condition is fufilled if y is unitary.) Then

$$|ya| = |a|$$

and $(yx, |a|)$ is the polar representation of ya.

We put

$$q := xx^*\,, \quad r := y^*y\,.$$

Then $q \in \Pr E$,

$$x^*q = x^*\,, \quad qx = x$$

(Corollary 4.1.2.22), and

$$qrq = q$$

(Corollary 4.2.7.6 a $\Rightarrow$ e). It follows

$$(yx)^*(yx) = x^*y^*yx = x^*rx = x^*qrqx = x^*qx = x^*x\,.$$

Hence $(yx)^*(yx)$ is the carrier of $|a|^{\cdot}$. Since

$$ya = yx|a|\,,$$

it follows that $(yx, |a|^{\cdot})$ is the polar representation of ya and

$$|ya| = |a|\,. \qquad \blacksquare$$

Corollary 4.4.3.7 (0) *Let $\mathbb{K} = \mathbb{R}$. Take $a \in F$, and let (x, b) be the polar representation of a. Then (using the notation in Theorem 4.4.1.2) $((x,0), \widetilde{(b,0)})$ is the polar representation of $\widetilde{(a,0)}$.*

By Proposition 4.3.6.1 a),b),

$$\widetilde{(a,0)} \in (\overset{\circ}{E})'\,, \quad \widetilde{(b,0)} \in (\overset{\circ}{E})'_+\,.$$

By Corollary 4.3.6.2 e),f),

$$\widetilde{(a,0)} = (x,0)\widetilde{(b,0)}$$

and $(x,0)^*(x,0)$ is the carrier of $\widetilde{(b,0)}$. Hence $((x,0), \widetilde{(b,0)})$ is the polar representation of $\widetilde{(a,0)}$. $\blacksquare$

Proposition 4.4.3.8 (0) *Let G be a closed C^*-subalgebra of E_F and $u : G \to E$ the inclusion map. By Theorem 4.4.2.20, $u'(F)$ is an involutive G-submodule of G' and the map*

$$G \longrightarrow (u'(F))', \quad x \longmapsto \langle x, \cdot \rangle | u'(F)$$

is an isometry (i.e. $u'(F)$ is a predual of G).

a) $a \in F_+ \Rightarrow u'a \in u'(F)_+$.

b) *Take $a \in F$ and let p be its left (right) carrier. If $p \in G$, then p is the left (right) carrier of $u'a$.*

c) *Take $a \in F$ and let $(x, |a|)$ be its polar representation. If $x \in G$, then $(x, u'|a|)$ is the polar representation of $u'a$.*

a) follows from Corollary 4.2.2.11.

b) Consider

$$A := \{q \in \Pr E \mid qa = a\} \quad (A := \{q \in \Pr E \mid aq = a\}).$$

p is the infimum of A (Proposition 4.3.3.2 b)). If $p \in G$, then p is the infimum of $A \cap \Pr G$. By Proposition 4.3.3.2 b) and Corollary 4.4.2.17, p is the left (right) carrier of ua.

c) x^*x is the carrier of $|a|$. By a), $u'|a|$ is positive and by b), x^*x is the carrier of $u'|a|$, i.e. $(x, u'|a|)$ is the polar representation of $u'a$ (Proposition 2.2.7.6). ■

Theorem 4.4.3.9 (0) *Take $a \in \operatorname{Re} F$ and let $(x, |a|)$ be the polar representation of a. Then x is selfadjoint and*

$$xa = ax, \quad x|a| = |a|x,$$

$$a^+ = x^+a = ax^+ = x^+|a| = |a|x^+ \in F_+,$$

$$a^- = -x^-a = -ax^- = x^-|a| = |a|x^- \in F_+,$$

$$a^+ + a^- = |a|,$$

$|x|$, x^+, and x^- are the carriers of $|a|$, a^+, and a^-, respectively (and belong therefore to $\Pr E$).

Step 1 $\mathbb{K} = \mathbb{C} \Rightarrow x$ selfadjoint

Since

$$a = a^*,$$

we see that

$$x^*x = xx^*$$

(Theorem 4.4.3.5 b)). Let G be a maximal Gelfand C^*-subalgebra of E containing x (Corollary 4.1.2.3). By Corollary 4.4.2.7, G is closed in E_F and $\sigma(G)$ is a Stonian space. Let $u : G \to E$ be the inclusion map. By Proposition 4.4.3.8 e), $u'a$ belongs to the predual $u'(F)$ of G and $(x, u'|a|)$ is the polar representation of $u'a$. By Theorem 4.4.1.8 c), $u'a$ is order continuous. By Example 1.7.2.14 c), x is selfadjoint.

Step 2 $\mathbb{K} = \mathbb{R} \Rightarrow x$ selfadjoint

By Corollary 4.4.3.7, $((x,0), \widetilde{(|a|,0)})$ is the polar representation of $\widetilde{(a,0)}$. Since $\widetilde{(a,0)}$ is selfadjoint it follows from Step 1 that $(x,0)$ is selfadjoint, i.e. x is selfadjoint.

Step 3 The other assertions

We deduce, successively, that

$$x|a| = a = a^* = |a|x,$$

$$x^+|a| = |a|x^+, \quad x^-|a| = |a|x^-$$

(Corollary 4.2.8.16),

$$\begin{cases} x^+a = x^+x|a| = x^+|a| = |a|x^+ = |a|xx^+ = ax^+, \\ -x^-a = -x^-x|a| = x^-|a| = |a|x^- = -|a|xx^- = -ax^- \end{cases}$$

(Proposition 4.2.2.9),

$$xa = |a| = |a|^* = ax,$$

$$(j_F a)^+ = (j_F x|a|)^+ = x^+ j_F(|a|) = j_F(x^+|a|) = j_F(x^+a),$$

$$(j_F a)^- = (j_F x|a|)^- = x^- j_F(|a|) = j_F(x^-|a|) = -j_F(x^- a)\,,$$

(Corollary 4.2.8.16, Proposition 2.2.7.5). By Proposition 4.2.2.9 and Theorem 4.4.3.5 b), $|x|$ is the carrier of $|a|$ and a and $x^+, x^- \in \Pr E$. It follows $a^+, a^- \in F$,

$$x^+ a^+ = (x^+)^2 a = x^+ a = a^+\,,$$

$$x^- a^- = -(x^-)^2 a = -x^- a = a^-\,,$$

$$a^+ + a^- = (x^+ - x^-)a = xa = x^2|a| = |a|\,.$$

Take $y \in E$ with

$$ya^+ = 0 \qquad (ya^- = 0)\,.$$

Then

$$yx^+ a = ya^+ = 0 \qquad (-yx^- a = ya^- = 0)$$

so that

$$yx^+ = yx^+|x| = 0 \qquad (yx^- = -yx^-|x| = 0)\,.$$

Hence x^+ and x^- are the carriers of a^+ and a^-, respectively. ■

Corollary 4.4.3.10 (0) *For every $a \in F$, the map*

$$E \longrightarrow F\,, \quad x \longmapsto ax \qquad (resp.\ xa)$$

is order continuos, i.e. $F = \dot{F}$ (Definition 4.3.1.7).

By Proposition 4.2.2.15 d) (and Corollary 4.3.6.2 e)), we may assume $\mathbb{K} = \mathbb{C}$. Let A be a nonempty downward directed set of E with infimum 0 and let $\mathfrak{F}$ be its lower section filter. We need to prove that

$$\lim_{x,\mathfrak{F}} ax = \lim_{x,\mathfrak{F}} xa = 0\,.$$

Since a is a linear combination of four positive elements of F (Theorem 4.4.3.9), we may assume a to be positive. By Theorem 4.4.1.8 c),

$$\inf_{x \in A} \langle a, x \rangle = 0$$

and the assertion follows from Corollary 4.3.1.6 a $\Rightarrow$ b & c. ■

Corollary 4.4.3.11 *Take* $a \in F$ *(resp.* $z \in E$*) and* $x, y \in \operatorname{Sn} E$ *such that*

$$xa = ay \quad (\textit{resp.}\ xz = zy)\,.$$

Then for every bounded Borel function f *on* $\sigma(x) \cup \sigma(y)$,

$$f(x)a = af(y) \qquad (\textit{resp.}\ f(x)z = zf(y))\,.$$

Let $\mathcal{B}$ denote the C^*-algebra of bounded Borel functions on $\sigma(x) \cup \sigma(y)$ and put

$$\widetilde{\mathcal{B}} := \{f \in \mathcal{B} \mid f(x)a = af(y) \quad (\text{resp. } f(x)z = zf(y))\}\,.$$

Let $(f_n)_{n\in\mathbb{N}}$ be an increasing (decreasing) sequence in $\widetilde{\mathcal{B}}$ with supremum (infimum) f in $\mathcal{B}$. By Corollary 4.4.3.10,

$$f(x)a = af(y)$$

(by Theorem 4.4.1.8 c),

$$\langle b, f(x)z\rangle = \langle zb, f(x)\rangle = \lim_{n\to\infty}\langle zb, f_n(x)\rangle =$$

$$= \lim_{n\to\infty}\langle b, f_n(x)z\rangle = \lim_{n\to\infty}\langle b, zf_n(y)\rangle = \lim_{n\to\infty}\langle bz, f_n(y)\rangle = \langle bz, f(y)\rangle = \langle b, zf(y)\rangle$$

for every $b \in F$, so that

$$f(x)z = zf(y)\,.$$

Hence $f \in \widetilde{\mathcal{B}}$. Since $\mathcal{C}(\sigma(x) \cup \sigma(y)) \subset \widetilde{\mathcal{B}}$ (Theorem 4.1.4.1), $\mathcal{B} = \widetilde{\mathcal{B}}$. ■

Proposition 4.4.3.12 *Take* $a \in F_+$ *and* $x \in \operatorname{Sn} E$. *If*

$$xa = -ax\,,$$

then

$$xa = ax = 0\,.$$

First suppose that x is selfadjoint. Then

$$x^+a = a(-x)^+ = ax^-$$

(Theorem 4.1.4.1, Theorem 4.2.2.9 b),d)). Let p and q be the carriers of x^+ and x^-, respectively (Theorem 4.4.1.8 j)). Since

$$pq = 0$$

(Proposition 4.3.3.15), we have that

$$x^+ap = ax^-p = ax^-qp = 0\,,$$

$$\langle a, x^+\rangle = \langle a, p1x^+\rangle = \langle x^+ap, 1\rangle = 0\,,$$

$$\|x^+a\|^2 \leq \langle a, x^+\rangle \|x^+\|\,\|a\| = 0$$

(Proposition 4.3.1.4),

$$x^+a = 0\,,$$

$$ax^- = 0\,,$$

$$x^-a = (ax^-)^* = 0\,,$$

$$xa = (x^+ - x^-)a = 0\,,$$

$$ax = (xa)^* = 0\,.$$

Now let x be normal. By Theorem 4.1.4.1,

$$(\mathrm{re}\, x)a = -a(\mathrm{re}\, x)\,, \quad (\mathrm{im}\, x)a = -a(\mathrm{im}\, x)\,.$$

By the above considerations,

$$(\mathrm{re}\, x)a = a(\mathrm{re}\, x) = 0\,, \quad (\mathrm{im}\, x)a = a(\mathrm{im}\, x) = 0\,,$$

so that

$$xa = ax = 0\,. \qquad \blacksquare$$

Corollary 4.4.3.13 *Let $a \in F_+$ and p the carrier of a (Corollary 4.4.2.17). The map*

$$p(\mathrm{Re}\, E)p \longrightarrow F\,, \quad x \longmapsto xa + ax$$

is injective.

Take $x \in p(\operatorname{Re} E)p$ such that

$$xa + ax = 0\,.$$

By Corollary 4.4.3.12,

$$xa = 0\,,$$

so that

$$x = xp = 0\,. \qquad \blacksquare$$

Corollary 4.4.3.14 *Let $a \in F_+$, p the carrier of a (Corollary 4.4.2.17), and*

$$L := \{b \in F_+ \mid b \leq a\}\,,$$

$$K := \{x \in pE_+^{\#}p \mid \tfrac{1}{2}(xa + ax) \in L\}\,.$$

Then K_F and L_E are compact and the map

$$K_F \longrightarrow L_E\,, \quad x \longmapsto \frac{1}{2}(xa + ax)$$

is a homeomorphism.

By Propositions 4.4.1.3 and 4.4.2.2., $pE_+^{\#}p$ is a compact set of E_F. The map

$$E_F \longrightarrow F_E\,, \quad x \longmapsto \frac{1}{2}(xa + ax)$$

is continuous (Corollary 4.4.2.10). Since F_+ is a closed set of F_E, L is also a closed set of F_E and therefore K is closed in $(pE_+^{\#}p)_F$. Hence K_F is compact.

Assume the map

$$K \longrightarrow L\,, \quad x \longmapsto \frac{1}{2}(xa + ax)$$

is not surjective. Then there is a

$$b \in L \backslash \{\frac{1}{2}(xa + ax) \mid x \in pE_+^{\#}p\}\,.$$

By Proposition 2.3.2.24, there is a $y \in \operatorname{Re} E$ such that

$$\sup_{x \in pE_+^{\#}p} \langle \frac{1}{2}(xa + ax), y\rangle < \langle b, y\rangle\,.$$

Then

$$\sup_{x\in pE_+^{\#}p}\langle\frac{1}{2}(xa+ax),pyp\rangle=\sup_{x\in pE_+^{\#}p}\langle\frac{1}{2}p(xa+ax)p,y\rangle=$$

$$=\sup_{x\in pE_+^{\#}p}\langle\frac{1}{2}(xa+ax),y\rangle<\langle b,y\rangle=\langle pbp,y\rangle=\langle b,pyp\rangle$$

(Corollary 4.4.2.18, Proposition 4.3.3.2 a)). If q denotes the carrier of $(pyp)^+$ (Theorem 4.4.1.8 j)), then $q\in pE_+^{\#}p$ (Example 4.3.4.2, Corollary 4.4.1.13) and therefore

$$\langle b,(pyp)^+\rangle\geq\langle b,pyp\rangle>\langle\frac{1}{2}(qa+aq),pyp\rangle=\langle a,\frac{1}{2}(pypq+qpyp)\rangle=\langle a,(pyp)^+\rangle$$

(Proposition 4.3.3.15). This contradicts the relation $b\leq a$. Hence the map

$$K\longrightarrow L\,,\quad x\longmapsto\frac{1}{2}(xa+ax)$$

is surjective. In particular, L_E is compact.

By Corollary 4.4.3.13, the map

$$K_F\longrightarrow L_E\,,\quad x\longmapsto\frac{1}{2}(xa+ax)$$

is injective, and so it is a homeomorphism. ■

Remark. $x\in pE_+^{\#}p$ does not imply

$$\frac{1}{2}(xa+ax)\geq 0$$

as one can see from the example in the remark to the next corollary.

Corollary 4.4.3.15 (Theorem of Radon–Nikodym) *Take $a\in F_+$. Let p be the carrier of a and take $b\in\operatorname{Re}F$ such that $|b|\leq a$. Then there is a unique $x\in p(\operatorname{Re}E)p$ such that*

$$b=\frac{1}{2}(xa+ax)\,.$$

We have $|x|\leq p$.

We have $b^+,\,b^-\in F_+$,

$$b^+\leq|b|\,,\quad b^-\leq|b|$$

(Theorem 4.4.3.9). By Corollary 4.4.3.14, there are $y,z\in pE_+^{\#}p$ such that

$$b^+ = \frac{1}{2}(ya + ay)\,, \quad b^- = \frac{1}{2}(za + az)\,.$$

Put

$$x := y - z\,.$$

Then $x \in p(\operatorname{Re} E^{\#})p$ and

$$\frac{1}{2}(xa + ax) = b^+ - b^- = b\,.$$

The uniqueness follows from Corollary 4.4.3.13. By Proposition 4.2.7.1 c $\Rightarrow$ a,

$$|x| \leq p\,. \qquad \blacksquare$$

Remark. In general,

$$\frac{1}{2}(xa + ax)^+ \neq \frac{1}{2}(x^+a + ax^+)\,.$$

Indeed, putting

$$a := \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\,, \quad x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,,$$

we obtain

$$x^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\,,$$

$$\frac{1}{2}(x^+a + ax^+) = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) =$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\,,$$

so

$$\frac{1}{2}(x^+a + ax^+)$$

is not even positive.

Corollary 4.4.3.16 (8) *Let* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$.

a) *The absolute convex hull of* $F_+^{\#}$ *is a* 0*-neighbourhood in* F *(in* $\operatorname{Re} F$*).*

b) *If x is an (involutive) linear form on F such that $x(F_+) \subset \mathbb{R}_+$, then $x \in E_+$.*

c) *For every $x \in E_+$,*

$$\|x\| = \sup_{a \in F_+^\#} \langle x, a \rangle .$$

d) *$E_+'^\#$ is the closure of $F_+^\#$ in E_E'.*

a) follows from Theorem 4.4.3.9.

b) By Theorem 4.4.3.9, $x(\operatorname{Re} F) \subset \mathbb{R}$ so that x is involutive (Corollary 2.3.1.23 b $\Rightarrow$ a). By a) and Corollary 1.7.1.8 b), x is continuous so that $x \in E$. By Corollary 4.4.1.5, $x \in E_+$.

c) Take $b \in \operatorname{Re} F^\#$. Then $b^+, b^- \in F_+^\#$ (Theorem 4.4.3.9), so that

$$-\langle x, b^- \rangle \leq \langle x, b \rangle \leq \langle x, b^+ \rangle$$

$$|\langle x, b \rangle| \leq \sup_{a \in F_+^\#} \langle x, a \rangle .$$

By Proposition 2.3.2.22 j),

$$\|x\| = \sup_{b \in \operatorname{Re} F^\#} |\langle x, b \rangle| = \sup_{a \in F_+^\#} \langle x, a \rangle .$$

d) Let $\overline{F_+^\#}$ be the closure of $F_+^\#$ in E_E'. It is contained in $E'^\#$ (Proposition 1.2.6.6). Take $x' \in E_+'^\# \backslash \overline{F_+^\#}$. We may assume $\|x'\| = 1$. By Proposition 2.3.2.25, there is an $x \in \operatorname{Re} E$ such that

$$\sup_{a \in F_+^\#} \langle x, a \rangle < \langle x, x' \rangle .$$

Put

$$y := \|x\| 1 + x \in E_+ .$$

By c),

$$\|y\| = \sup_{a \in F_+^\#} \langle y, a \rangle \leq \|x\| + \sup_{a \in F_+^\#} \langle x, a \rangle <$$

$$< \|x\| + \langle x, x' \rangle \leq \langle \|x\| 1 + x, x' \rangle = \langle y, x' \rangle \leq \|y\|$$

(Corollary 2.3.4.7), which is a contradiction. Hence $\overline{F_+^\#} = E_+'^\#$. ■

Proposition 4.4.3.17 *Put*

$$\widetilde{x} : F \longrightarrow F\,, \quad a \longmapsto xa$$

for every $x \in E$.

a) $\widetilde{x} \in \mathcal{L}(F)$ *and* $\|\widetilde{x}\| = \|x\|$ *for every* $x \in E$.

b) *The map*

$$E \longrightarrow \mathcal{L}(F)\,, \quad x \longmapsto \widetilde{x}$$

is an algebra homomorphism preserving the norms. We identify E *with* $\{\widetilde{x} | x \in E\}$ *and denote by* $\mathfrak{T}$ *the topology induced on* E *by the topology of pointwise convergence of* $\mathcal{L}(F)$.

c) $\mathfrak{T}$ *is finer (coarser) then the topology of* E_F *(of* E*).*

d) *The map*

$$E^{\#} \times E \longrightarrow E\,, \quad (x, y) \longmapsto xy$$

is continuous with respect to $\mathfrak{T}$.

e) $\Pr E$ *is a closed set of* E *with respect to* $\mathfrak{T}$.

f) *For every* $f \in \mathcal{C}([-1, 1])$, *the map*

$$\operatorname{Re} E^{\#} \longrightarrow E\,, \quad x \longmapsto f(x)$$

is continuous with respect to $\mathfrak{T}$.

g) *Let* A *be an upward (downward) directed set of* E, x *its supremum (infimum), and* $\mathfrak{F}$ *its upper (lower) section filter. Then* $\mathfrak{F}$ *converges to* x *with respect to* $\mathfrak{T}$.

h) *Every linear map of* E *into a normed space is order continuous, whenever it is continuous with respect to* $\mathfrak{T}$.

i) *For every* $a \in F$, *the map*

$$E \longrightarrow F\,, \quad x \longmapsto xa \quad (\textit{resp. } ax)$$

is continuous with respect to $\mathfrak{T}$.

a) $\widetilde{x}$ is linear and

$$\|\widetilde{x}a\| = \|xa\| \leq \|x\| \, \|a\|$$

for every $a \in F$. Hence $\widetilde{x} \in \mathcal{L}(F)$ and $\|\widetilde{x}\| \leq \|x\|$. On the other side,

$$\|x\| = \sup_{a \in F^\#} |\langle x, a \rangle| = \sup_{a \in F^\#} |\langle 1, xa \rangle| = \sup_{a \in F^\#} \|xa\| = \sup_{a \in F^\#} \|\widetilde{x}a\| \leq \|\widetilde{x}\|$$

so that $\|\widetilde{x}\| = \|x\|$.

b) Take $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. Then for every $a \in F$,

$$\widetilde{\alpha x + \beta y}a = (\alpha x + \beta y)a = \alpha xa + \beta ya = \alpha \widetilde{x}a + \beta \widetilde{y}a = (\alpha \widetilde{x} + \beta \widetilde{y})a,$$

$$\widetilde{xy}a = (xy)a = x(ya) = \widetilde{x}(ya) = \widetilde{x}\widetilde{y}a,$$

so that

$$\widetilde{\alpha x + \beta y} = \alpha \widetilde{x} + \beta \widetilde{y}, \quad \widetilde{xy} = \widetilde{x}\widetilde{y}.$$

Hence the map of b) is an algebra homomorphism. By a), it preserves the norms.

c) By a), for $a \in F$ and $x \in E$,

$$|\langle x, a \rangle| = |\langle 1, xa \rangle| \leq \|xa\| = \|\widetilde{x}a\| \leq \|\widetilde{x}\| \|a\| = \|x\| \, \|a\|$$

which proves the assertion.

d) follows from b) and Proposition 1.2.6.7.

e) By c) and Proposition 4.4.1.3, $\operatorname{Re} E$ is closed with respect to $\mathfrak{T}$ and by d), $\{x \in E^\# | x^2 = x\}$ is also closed with respect to $\mathfrak{T}$. Hence $\operatorname{Pr} E$ is closed with respect to $\mathfrak{T}$.

f) By d), the assertion holds if f is a polynomial. Since f is the uniform limit of polynomials, the assertion holds, by b), also for f.

g) follows from Corollary 4.4.3.10.

h) follows from g).

i) is obvious. ■

4.4.4 W^*-Homomorphisms

Proposition 4.4.4.1 (0) *Take* $x', y' \in E'_+$ *with*

$$x'(1) < y'(1)\,.$$

If

$$\sup_{p\in A} x'(p) = x'\left(\bigvee_{p\in A} p\right)\,, \quad \sup_{p\in A} y'(p) = y'\left(\bigvee_{p\in A} p\right)\,,$$

for every well-ordered set A *of* $\Pr E$, *then there is a* $p \in \Pr E\backslash\{0\}$ *with*

$$x'(pxp) \leq y'(pxp)$$

for every $x \in E_+$.

$\Pr E$ is order complete (Theorem 4.4.1.8 i)), so by our hypothesis (and by Zorn's Lemma), there is a maximal element q of the set

$$\{r \in \Pr E \mid x'(r) \geq y'(r)\}\,.$$

Put

$$p := 1 - q\,.$$

It is obvious that $p \neq 0$. Take $r \in \Pr E$ with $r \leq p$. Then

$$r + q = r + (1 - p) \in \Pr E$$

(Corollary 4.2.7.7), so that

$$x'(r) \leq y'(r)\,.$$

Take $\varepsilon > 0$. By Proposition 4.4.2.2, pEp is a W^*-algebra with p as unit. By Theorem 4.4.1.8 f), there is a finite family $((\alpha_\iota, p_\iota))_{\iota\in I}$ in $\mathbb{R}_+ \times \Pr(pEp)$, such that

$$\left\| pxp - \sum_{\iota\in I} \alpha_\iota p_\iota \right\| < \frac{\varepsilon}{\|y' - x'\|}\,.$$

We have $p_\iota \in \Pr E$ and $p_\iota \leq p$ for every $\iota \in I$ (Corollary 4.2.7.6 e $\Rightarrow$ a). Thus

$$\left| \langle pxp, y' - x'\rangle - \langle \sum_{\iota\in I} \alpha_\iota p_\iota, y' - x'\rangle \right| =$$

$$= \left| \left\langle pxp - \sum_{\iota \in I} \alpha_\iota p_\iota, y' - x' \right\rangle \right| \leq \left\| \left\langle pxp - \sum_{\iota \in I} \alpha_\iota p_\iota \right\rangle \right\| \, \|y' - x'\| < \varepsilon ,$$

$$y'(pxp) - x'(pxp) = \langle pxp, y' - x' \rangle \geq \left\langle \sum_{\iota \in I} \alpha_\iota p_\iota, y' - x' \right\rangle - \varepsilon =$$

$$= \sum_{\iota \in I} \alpha_\iota \langle p_\iota, y' - x' \rangle - \varepsilon \geq -\varepsilon .$$

Since ε is arbitrary,

$$x'(pxp) \leq y'(pxp) .$$ ■

Theorem 4.4.4.2 (0) *Given $x' \in E'_+$, the following are equivalent:*

a) $x' \in F$.

b) x' *is order continuous.*

c) *Given any well-ordered set A of* $\Pr E$,

$$\sup_{p \in A} x'(p) = x' \left(\bigvee_{p \in A} p \right) .$$

a $\Rightarrow$ b follows from Theorem 4.4.1.8 c).

b $\Rightarrow$ c follows from Theorem 4.4.1.8 i).

c $\Rightarrow$ a. Consider

$$A := \{ p \in \Pr E \mid px' \in F \} .$$

Let B be a nonempty, well-ordered set of A, $\mathfrak{F}$ the upper section filter of B, and p the supremum of B (Theorem 4.4.1.8 i)). By c),

$$\lim_{q, \mathfrak{F}} \langle p - q, x' \rangle = 0 ,$$

so that

$$\lim_{q, \mathfrak{F}} (p - q)x' = 0$$

(Corollary 4.3.1.6 a $\Rightarrow$ b),

$$px' = \lim_{q, \mathfrak{F}} qx' \in F .$$

By Zorn's Lemma, A has a maximal element p.

Assume that $p \neq 1$. Then there is a $y' \in F_+$ with

$$y'(1-p) > x'(1-p)$$

(Corollary 4.4.1.6 b)). By a $\Rightarrow$ c, Proposition 4.4.2.2 and Proposition 4.4.4.1, there is a $q \in \Pr E \backslash \{0\}$ with $q \leq 1-p$ and

$$x'(qxq) \leq y'(qxq)$$

for every $x \in E_+$. We deduce that

$$|\langle x, qx' \rangle|^2 = |x'(xq)|^2 \leq x'(qx^*xq)x'(1) \leq y'(qx^*xq)x'(1) = \langle x^*x, qy'q \rangle x'(1)$$

for every $x \in E$ (Proposition 2.3.4.6 c)). Endow $E^\#$ with the topology of uniform convergence on the weakly compact convex sets of F. Let $\mathfrak{V}$ be the neighbourhood filter of 0 in $E^\#$. Since $qy'q \in F$ (Corollary 4.4.2.10),

$$\lim_{x,\mathfrak{V}} \langle x^*x, qy'q \rangle = 0$$

(Corollary 4.4.2.12). Thus

$$\lim_{x,\mathfrak{V}} \langle x, qx' \rangle = 0\,.$$

Hence $(qx')|E^\#$ is continuous at 0. By Proposition 1.3.7.9 a $\Rightarrow$ d, $qx' \in \operatorname{Im} j_F$. Then $(p+q)x' \in F$, i.e. $p+q \in A$ (Corollary 4.2.7.7), and this contradicts the maximality of p. Hence $p = 1$ and $x' \in F$. ■

Corollary 4.4.4.3 (0) *Let* $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). *Given* $x' \in E'$ ($x' \in \operatorname{Re} E'$) *the following are equivalent:*

a) $x' \in F$.

b) $x' \in E^\pi$.

These conditions imply the following equivalent assertions:

c) $x'|G \in G^\pi$ *for every maximal Gelfand* C^**-subalgebra* G *of* E.

d) *If* A *is a commutative downward directed set of* $\Pr E$ *with infimum* 0 *and* $\mathfrak{F}$ *denotes its lower section filter, then* $x'(\mathfrak{F})$ *converges to* 0.

a $\Rightarrow$ b. Assume first x' selfadjoint. By Theorem 4.4.3.9, $x'^+, x'^- \in F$ and by Theorem 4.4.4.2 a $\Rightarrow$ b,

$$x' = x'^+ - x'^- \in E^\pi .$$

If x' is arbitrary, then $\operatorname{re} x'$, $\operatorname{im} x' \in F$, so that by the above considerations,

$$x' = \operatorname{re} x' + i \operatorname{im} x' \in E^\pi .$$

b $\Rightarrow$ a follows from Theorem 4.4.4.2 b $\Rightarrow$ a.

b $\Rightarrow$ c follows from Corollary 4.4.2.7.

c $\Rightarrow$ d. Since A is commutative, there is a maximal Gelfand C^*-subalgebra G of E containing A (Corollary 4.1.2.3). Then 0 is the infimum of A in G (Theorem 4.4.1.8 c),i)). Thus $x'(\mathfrak{F})$ converges to 0.

d $\Rightarrow$ c. There is a Radon measure μ on $\sigma(G)$ such that

$$x'(x) = \int \hat{x} d\mu$$

for every $x \in G$. Let K be a nowhere dense compact set of $\sigma(G)$ and $\mathfrak{U}$ the set of clopen sets of $\sigma(G)$ containing K. $\mathfrak{U}$ is downward directed and by Corollary 4.4.2.7,

$$\bigcap_{U \in \mathfrak{U}} U = K .$$

By d), $\mu(K) = 0$. Since K is arbitrary, the meager sets of $\sigma(G)$ are μ-null sets. By Example 1.7.2.14 a), $\mu \in \mathcal{C}(\sigma(G))^\pi$, i.e. $x'|G \in G^\pi$. ■

Remark. The implication c $\Rightarrow$ a will be proved in Theorem 6.3.2.2 $j_4 \Rightarrow j_2$.

Corollary 4.4.4.4 (Dixmier, 1953) (0) *Let* $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$) *and let* G *be a* W^**-algebra. Then there is a unique closed vector subspace (a unique involutive* G*-submodule)* H *of* G' *such that the map*

$$G \longrightarrow H', \quad x \longmapsto \langle x, \cdot \rangle | H$$

is an isometry. H *is an involutive* G*-submodule of* G' *and the above map is an isometry of involutive* G*-modules.* H *is called* ***the predual of*** G *and will be denoted by* $\ddot{G}$.

The existence of H is simply a restatement of the definition of a W^*-algebra. For $\mathbb{K} = \mathbb{C}$, the uniqueness follows immediately from Corollary 4.4.4.3 a $\Leftrightarrow$ b.

We prove the uniqueness when $\mathbb{K} = \mathbb{R}$. Let H_1, H_2 be two involutive G-submodules of G' enjoying the above property. By Theorem 4.3.6.4 (and using

the notation from there), $\widetilde{H}_1, \widetilde{H}_2$ are closed vector subspaces of $(\overset{\circ}{G})'$ and the map

$$\overset{\circ}{G} \longrightarrow (\widetilde{H}_k)', \quad z \longmapsto \langle z, \cdot \rangle | \widetilde{H}_k$$

is an isometry for $k \in \{1, 2\}$. By the uniqueness in the complex case (and Theorem 4.4.1.2), $\widetilde{H}_1 = \widetilde{H}_2$. Take $x' \in H_1$. Then there is a $(y', z') \in \overset{\circ}{H}_2$ with

$$\widetilde{(x', 0)} = \widetilde{(y', z')}.$$

We get

$$x'(x) = y'(x) + iz'(x)$$

for every $x \in G$. Hence $x' = y' \in H_2$, $H_1 \subset H_2$, and so $H_1 = H_2$.

By Corollary 4.4.2.10, H is an involutive G–submodule of G' and the map

$$G \longrightarrow H', \quad x \longmapsto \langle x, \cdot \rangle | H$$

is an isometry of involutive E–modules. ■

Remark. a) We occasionally identify E canonically with $(\ddot{E})'$ via the above isometry.

b) By the above corollary, the predual of a W^*–algebra is unique. But the duals of different C^*–algebras may "coincide". In fact, if T is a countable compact space, then $\mathcal{C}(T)'$ may be identified with $\ell^1(T)$. Hence if S and T are countable compact spaces then there is an involutive isometry

$$u : \mathcal{C}(S)' \longrightarrow \mathcal{C}(T)'$$

such that

$$u(\mathcal{C}(S)'_+) = \mathcal{C}(T)'_+.$$

Definition 4.4.4.5 (0) *Let G be a W^*–algebra and $u : E \to G$ an involutive algebra homomorphism. u is called a* ***W^*–homomorphism*** *if $u'(\ddot{G}) \subset \ddot{E}$. u is called an* ***isometry*** *or an* ***isomorphism of W^*–algebras*** *if it is an isomorphism of C^*–algebras (in which case it is automatically a W^*–homomorphism).*

A ***(unital) W^*–subalgebra of*** *E is a (unital) C^*–subalgebra of E which is closed in $E_{\ddot{E}}$. For $A \subset E$, the smallest (unital) W^*–subalgebra of E containing A is called* ***the (unital) W^*–subalgebra of E generated by*** *A.*

If E_1, E_2, E_3 are W^*-algebras and $u : E_1 \to E_2$, $v : E_2 \to E_3$ are W^*-homomorphisms, then $v \circ u$ is also a W^*-homomorphism.

If G is a C^*-algebra and $u : E \to G$ a bijective involutive algebra homomorphism, then u is an isometry (Corollary 4.1.1.21), so that G is a W^*-algebra and u is an isometry of W^*-algebras.

Proposition 4.4.4.6 (0) *Let G be a W^*-algebra and $u : E \to G$ an involutive algebra homomorphism. Then u is a W^*-homomorphism iff the map*

$$E_{\ddot{E}} \longrightarrow G_{\ddot{G}}, \quad x \longmapsto ux$$

is continuous.

Take $y' \in \ddot{G}$. We have $u'y' = y' \circ u$. Hence if u is a W^*-homomorphism, then $u'y' \in \ddot{E}$ and therefore the map

$$E_{\ddot{E}} \longrightarrow \mathbb{K}, \quad x \longmapsto \langle ux, y' \rangle = \langle x, u'y' \rangle$$

is continuous. Thus the map

$$E_{\ddot{E}} \longrightarrow G_{\ddot{G}}, \quad x \longmapsto ux$$

is continuous. Conversely, if this map is continuous, then the map

$$E_{\ddot{E}} \longrightarrow \mathbb{K}, \quad x \longmapsto \langle x, u'y' \rangle = \langle ux, y' \rangle$$

is continuous and therefore $u'y' \in \ddot{E}$ (Corollary 1.2.6.5). Thus $u'(\ddot{G}) \subset \ddot{E}$ and u is a W^*-homomorphism. ∎

Proposition 4.4.4.7 (0) *Let E, G be real W^*-algebras and $u : E \to G$ an involutive algebra homomorphism. Put*

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow \overset{\circ}{G}, \quad (x, y) \longmapsto (ux, uy).$$

Then u is a W^*-homomorphism iff $\overset{\circ}{u}$ is a W^*-homomorphism. If H is a C^*-subalgebra of E, then H is a W^*-subalgebra of E iff $\overset{\circ}{H}$ is a W^*-subalgebra of $\overset{\circ}{E}$.

By Proposition 2.3.1.41, $\overset{\circ}{u}$ is an involutive algebra homomorphism and therefore continuous (Corollary 4.1.1.20). We identify $(\overset{\circ}{E})'$ with $\overset{\circ}{\widehat{E'}}$ and $(\overset{\circ}{G})'$ with $\overset{\circ}{\widehat{G'}}$ as in Corollary 4.3.6.2 b). Take $(x', y') \in \overset{\circ}{\ddot{G}}$. By Proposition 4.3.6.6,

$$(\overset{\circ}{u})'((x',y')) = (u'x', u'y')\,.$$

By Theorem 4.3.6.4 b), $\overset{\circ\circ}{E}$ (resp. $\overset{\circ\circ}{G}$) is identified with $\overset{\circ}{\ddot{E}}$ (resp. $\overset{\circ}{\ddot{G}}$). Thus

$$(\overset{\circ}{u})'(\overset{\circ\circ}{G}) \subset \overset{\circ\circ}{E} \Longleftrightarrow u'(\ddot{G}) \subset \ddot{E}$$

i.e. u is a W^*-homomorphism iff $\overset{\circ}{u}$ is a W^*-homomorphism.

The final assertion follows from Theorem 4.4.1.2. ■

Corollary 4.4.4.8 (0) *Let G be a W^*-algebra, $u : E \to G$ a W^*-homomorphism, and*

$$v : \ddot{G} \longrightarrow \ddot{E}\,, \quad y' \longmapsto u'y'\,.$$

a) *For every $y \in \operatorname{Im} u$, there is an $x \in \overset{-1}{u}(y)$ with $\|x\| = \|y\|$.*

b) *If H is a W^*-subalgebra of E, then $u(H)$ is a W^*-subalgebra of G.*

c) *If we canonically identify E with $(\ddot{E})'$ and G with $(\ddot{G})'$, then v is involutive,*

$$u = v'\,, \quad \operatorname{Im} v = {}^\circ(\operatorname{Ker} u)\,, \quad \operatorname{Im} u = (\operatorname{Ker} v)^\circ\,.$$

v is called ***the pretranspose of*** *u.*

d) *If A is a downward directed set of E with infimum 0, then 0 is the infimum of $u(A)$.*

a) $\operatorname{Ker} u$ is a closed ideal of E_F (Proposition 4.4.4.6). Let $q : E \to E/\operatorname{Ker} u$ be the quotient map and $w : E/\operatorname{Ker} u \to \operatorname{Im} u$ the factorization of u through $E/\operatorname{Ker} u$. Since w is an isometry of C^*-algebras (Theorem 4.2.6.6), we have

$$\|y\| = \|w^{-1}y\|\,.$$

By Corollary 4.4.2.16, there is an $x \in E$ with

$$qx = w^{-1}y\,, \quad \|x\| = \|w^{-1}y\|\,.$$

We conclude that

$$x \in \overset{-1}{u}(y)\,, \quad \|x\| = \|y\|\,.$$

b) By a),

$$u(H)^{\#} = u(H^{\#}) .$$

Since $H^{\#}_{\ddot{E}}$ is compact (Alaoglu–Bourbaki Theorem) and since the map

$$E_{\ddot{E}} \longrightarrow G_{\ddot{G}} , \quad x \longmapsto ux$$

is continuous (Proposition 4.4.4.6), it follows that $u(H)^{\#}_{\ddot{G}}$ is compact. Hence $u(H)$ is a closed set of $G_{\ddot{G}}$ (Corollary 1.3.7.5), i.e. it is a W^*-subalgebra of G (Theorem 4.2.6.6).

c) By Proposition 2.3.2.22 d), u' is involutive, and so v is also involutive. The relation $u = v'$ is obvious. By Theorem 4.2.6.6, $\operatorname{Im} u$ is closed and the algebraic isomorphism $E/\operatorname{Ker} u \to \operatorname{Im} u$ associated to u is an isometry of C^*-algebras. By Proposition 1.4.2.11, $\operatorname{Im} v$ is closed, so that by Proposition 1.3.5.8 and Proposition 1.4.2.9 b),

$$\operatorname{Im} u = (\operatorname{Ker} v)^{\circ} , \quad \operatorname{Im} v = {}^{\circ}(\operatorname{Ker} u) .$$

d) Let $\mathfrak{F}$ be the lower section filter of A and take $y' \in \ddot{G}$. Then $u'y' \in \ddot{E}$, so that

$$\lim_{x,\mathfrak{F}} \langle ux, y' \rangle = \lim_{x,\mathfrak{F}} \langle x, u'y' \rangle = 0$$

(Theorem 4.4.1.8 c)), and so $u(\mathfrak{F})$ converges to 0 in $G_{\ddot{G}}$. By Theorem 4.4.1.8 c), 0 is the infimum of $u(A)$. ■

Corollary 4.4.4.9 (0) *Let G be a C^*-subalgebra of E and $u : G \to E$ the inclusion map. Then the following are equivalent:*

a) *G is a W^*-subalgebra of E.*

b) *G is a W^*-algebra and u is a W^*-homomorphism.*

If these conditions are fulfilled, then $\ddot{G} = u'(\ddot{E})$ and the unit of G is an orthogonal projection in E.

First suppose that *a)* holds. By Theorem 4.4.2.20 (and Corollary 4.4.4.4), G is a W^*-algebra and $\ddot{G} = u'(\ddot{E})$. Hence u is a W^*-homomorphism. The unit of G is an orthogonal projection of E since it is idempontent and selfadjoint.

b $\Rightarrow$ a follows from Proposition 4.4.4.8 b). ■

Remark. G can be a W^*-algebra without being a W^*-subalgebra of E. Indeed, let T be the Stone–Čech compactification of $\mathbb{N}$ and for every $x \in \ell^{\infty}$ let $\tilde{x}$ be its continuous extension on T. Then

$$G := \{\widetilde{x} | x \in \ell^\infty\}$$

is a W^*-algebra and a C^*-subalgebra of $\ell^\infty(T)$. Since it is dense in $\ell^\infty(T)_{\ell^1(T)}$, it is not a W^*-subalgebra of $\ell^\infty(T)$.

Corollary 4.4.4.10 (0) *If G is a W^*-subalgebra of E and H is a C^*-subalgebra of G, then H is a W^*-subalgebra of E iff it is a W^*-subalgebra of G.*

Let $u : G \to E$ and $v : H \to G$ be the inclusion maps. By Corollary 4.4.4.9 a $\Rightarrow$ b, G and H are W^*-algebras and u is a W^*-homomorphism. Moreover, if H is a W^*-subalgebra of G (resp. E), then v (resp. $u \circ v$) is a W^*-homomorphism. So, if H is a W^*-subalgebra of G, then $u \circ v$ is a W^*-homomorphism and by Corollary 4.4.4.9 b $\Rightarrow$ a, H is a W^*-subalgebra of E.

Now suppose that H is a W^*-subalgebra of E. By the last assertion of Corollary 4.4.4.9, $\ddot{G} = u'(\ddot{E})$. Thus

$$v'(\ddot{G}) = v'(u'(\ddot{E})) \;=\; (u \circ v)'(\ddot{E}) \subset \ddot{H}.$$

Hence v is a W^*-homomorphism and by Corollary 4.4.4.9 b $\Rightarrow$ a, H is a W^*-subalgebra of G. ■

Proposition 4.4.4.11 (0) *Let G be a unital W^*-subalgebra of E and $x \in G$.*

a) *The right and left carriers of x with respect to E (Theorem 4.4.1.8 j)) belong to G and are the right and left carriers of x with respect to G.*

b) *The polar representation of x in E and G (Theorem 4.4.3.1) coincide.*

c) *For every $A \subset \Pr G$, the supremum and the infimum of A in $\Pr E$ (Theorem 4.4.1.8 i)) belong to G.*

a) By Theorem 4.4.1.8 d), G is an order faithful set of E and the assertion follows from Corollary 4.3.3.12 (and Theorem 4.4.1.8 c)).

b) follows from a) and Theorem 4.4.3.1.

c) By Proposition 4.2.7.14, it is sufficient to prove the assertion for the supremum. Moreover, we may assume A nonempty.

Case 1 A finite

Put

$$x := \frac{1}{\operatorname{Card} A} \sum_{p \in A} p$$

and denote by q the supremum of A in $\operatorname{Pr} E$. $(x^{\frac{1}{k}})_{k \in \mathbb{N}}$ is an increasing sequence in G. By Proposition 4.2.7.21, q is the supremum of $(x^{\frac{1}{k}})_{k \in \mathbb{N}}$ in E. Since G is an order faithful set of E (Theorem 4.4.1.8 d)), it follows $q \in G$.

Case 2 A infinite

The assertion follows from Case 1 since G and $\operatorname{Pr} E$ are order faithful sets of E (Theorem 4.4.1.8 d),i)). ■

Corollary 4.4.4.12 (0)

a) *If G is an involutive subalgebra of E, then its closure in E_F is the W^*-subalgebra of E generated by G.*

b) *Every maximal Gelfand C^*-subalgebra of E is a W^*-subalgebra of E. In particular, the W^*-subalgebra of E generated by a self-normal element of E is a Gelfand W^*-algebra and the maximal Gelfand C^*-subalgebras of E and the maximal Gelfand W^*-subalgebras of E coincide.*

c) *For every $p \in \operatorname{Pr} E$, pEp is a hereditary W^*-subalgebra of E and every hereditary W^*-subalgebra of E is of this form. Moreover, pFp is an involutive pEp-submodule of F and the map*

$$pEp \longrightarrow (pFp)', \quad x \longmapsto \langle x, \cdot \rangle | pFp$$

is an isometry of involutive pEp-modules, so that pFp may be identified with $\widetilde{pEp}$.

d) *For every $p \in E^c \cap \operatorname{Pr} E$, pEp is a closed ideal of E_F and every closed ideal of E_F is of this form. Moreover, E/pEp is a W^*-algebra, the quotient map $q : E \longrightarrow E/pEp$ is a W^*-homomorphism, and $q|(1-p)E(1-p)$ is an isometry of W^*-algebras, so that $(1-p)F(1-p)$ may be identified with $\widetilde{E/pEp}$.*

e) *If A is a subset of E such that*

$$A \subset A^* \cup \operatorname{Sn} E,$$

then A^c is a unital W^*-subalgebra of E. If G is the W^*-subalgebra of E generated by A, then

$$G \subset A^{cc}, \quad A^c = G^c.$$

f) *If E is simple and $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$) then $E^c = \mathbb{C}1$ (E^c is isomorphic to $\mathbb{R}$ or $\mathbb{C}$).*

g) *If E is a Gelfand W^*-algebra then*

$$F = \{x' \in E' | x' \text{ is order continuous}\}.$$

h) *An ideal of E is closed in E_F iff it has a unit.*

a) follows from Corollary 4.4.2.9.

b) By Corollary 4.4.2.7, every maximal Gelfand C^*-subalgebra of E is closed in E_F. Hence the assertion follows from a).

c) By Proposition 4.4.2.2, the map

$$\varphi : E_F \longrightarrow E_F, \quad x \longmapsto pxp - x$$

is continuous. It follows from

$$pEp = \overset{-1}{\varphi}(0)$$

that pEp is closed in E_F. By Example 4.3.4.2, pEp is a hereditary W^*-subalgebra of E. Every hereditary W^*-subalgebra of E is of this form since its unit is an orthogonal projection in E.

It is obvious that pFp is an involutive pEp-submodule of F. By Corollary 4.4.2.10, the map of c) is a homomorphism of involutive pEp-modules. For every $x \in pEp$,

$$\|x\| = \sup_{a \in F^\#} |\langle x, a\rangle| = \sup_{a \in F^\#} |\langle pxp, a\rangle| =$$

$$= \sup_{a \in F^\#} |\langle x, pap\rangle| = \sup_{a \in (pFp)^\#} |\langle x, a\rangle| = \|\langle x, \cdot\rangle | pFp\|,$$

which shows (by Hahn–Banach Theorem) that this map is an isometry.

d) It is obvious that pEp is an ideal of E. By c), it is closed in E_F. Let now G be a closed ideal of E_F. By Proposition 4.3.4.5 e), G is a hereditary W^*-subalgebra of E. By c), there is a $p \in \Pr E$, such that

$$G = pEp\,.$$

Take $x \in E$. Then

$$px, xp \in G = pEp\,,$$

so that

$$px = pxp = xp\,.$$

Hence $p \in E^c$. Moreover,

$$x - (1 - p)x(1 - p) \in G$$

and therefore

$$qx = q(1 - p)x(1 - p)\,.$$

Thus $q|(1-p)E(1-p)$ is surjective. From

$$q(1 - p)x(1 - p) = 0$$

we get that

$$(1 - p)x(1 - p) \in G\,,$$

$$(1 - p)x(1 - p) = (1 - p)x(1 - p)p = 0\,.$$

Hence $q|(1-p)E(1-p)$ is injective. By Theorem 4.2.6.6, $q|(1-p)E(1-p)$ is an isometry of C^*–algebras. Hence E/G is a W^*–algebra and $q|(1-p)E(1-p)$ is an isometry of W^*–algebras. By c), $(1-p)F(1-p)$ may be identified with $\widetilde{E/G}$.

e) By Corollary 4.1.4.2 b), A^c is a unital C^*–subalgebra of E. By Corollary 4.4.2.5, A^c is closed in E_F and so A^c is a unital W^*–subalgebra of E.

By the above considerations, A^{cc} is a W^*–subalgebra of E. Since $A \subset A^{cc}$, we deduce successively that

$$A \subset G \subset A^{cc}\,,$$

$$A^c = A^{ccc} \subset G^c \subset A^c\,,$$

$$A^c = G^c\,.$$

f) By e), E^c is a W^*-subalgebra of E. By d), $E^c \cap \Pr E = \{0,1\}$ so that $E^c = \mathbb{C}1$ ($\operatorname{Re} E^c = \mathbb{R}1$) (Theorem 4.4.1.8 g)). By Theorem 4.2.8.24, E^c is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$ in the real case.

g) follows from Corollary 4.3.4.3 c $\Leftrightarrow$ d (and Proposition 4.2.7.17).

h) Let G be an ideal of E having a unit p. Then $G = Ep$, so that G is closed in E_F. The converse implication was proved in d). ■

Remark. The reverse implication in f) does not hold (Corollaries 6.1.7.14 b), 5.3.2.14, and 3.1.1.13).

Proposition 4.4.4.13 *Let E, G be complex W^*-algebras and $u : E \to G$ an involutive algebra homomorphism. Then the following are equivalent:*

a) *u is a W^*-homomorphism.*

b) $u'(\ddot{G}_+) \subset \ddot{E}_+$.

c) $u'(\ddot{G}_+) \subset \ddot{E}$.

a $\Rightarrow$ b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By Theorem 4.4.3.9,

$$\operatorname{Re} \ddot{G} = \ddot{G}_+ - \ddot{G}_+ .$$

Hence by c),

$$u'(\operatorname{Re} \ddot{G}) \subset u(\ddot{G}_+) - u(\ddot{G}_+) \subset \ddot{E} .$$

It follows that

$$u'(\ddot{G}) \subset \ddot{E} ,$$

i.e. u is a W^*-homomorphism. ■

Corollary 4.4.4.14 *Let E, G be complex W^*-algebras and $u : E \to G$ a homomorphism of involutive algebras. Then the following are equivalent:*

a) *u is a W^*-homomorphism.*

b) *(resp. c)) If A is a downward directed (and commutative) set of E_+ with infimum 0, then 0 is the infimum of $u(A)$.*

d) *If A is a well-ordered set of $\Pr E$ with supremum p, then up is the supremum of $u(A)$.*

a $\Rightarrow$ b follows from Proposition 4.4.4.8 d).

b $\Rightarrow$ c $\Rightarrow$ d is trivial.

d $\Rightarrow$ a. Take $y' \in \ddot{G}_+$. Let A be a well-ordered set of $\Pr E$ with supremum p and let $\mathfrak{F}$ be its upper section filter. By d) and Theorem 4.4.1.8 c), i), $y' \circ u(\mathfrak{F})$ converges to $y'(up)$. By Theorem 4.4.4.2 c $\Rightarrow$ a, $u'y' \in \ddot{E}$, i.e. $u'(\ddot{G}_+) \subset \ddot{E}$. By Proposition 4.4.4.13 c $\Rightarrow$ a, u is a W^*-homomorphism. ■

Corollary 4.4.4.15 *Assume* $\mathbb{K} = \mathbb{C}$. *Let* G *be a* C^**-subalgebra of* E. *Then the following are equivalent:*

a) G *is a* W^**-subalgebra of* E.

b) G *is a* W^**-algebra and an order faithful set of* E.

a $\Rightarrow$ b follows from Theorem 4.4.1.8 d).

b $\Rightarrow$ a follows from Corollary 4.4.4.14 b $\Rightarrow$ a and Corollary 4.4.4.9 b $\Rightarrow$ a. ■

Proposition 4.4.4.16 *Let* G *be a* W^**-algebra,* $u : E \to G$ *a* W^**-homomorphism,* $v : \ddot{G} \to \ddot{E}$ *the pretranspose of* u, $q : E \to E/\operatorname{Ker} u$, $r : \ddot{G} \to \ddot{G}/\operatorname{Ker} v$ *the quotient maps,* $i : \operatorname{Im} u \to G$, $j : \operatorname{Im} v \to \ddot{E}$ *the inclusion maps, and* $\overline{u} : E/\operatorname{Ker} u \to \operatorname{Im} u$, $\overline{v} : \ddot{G}/\operatorname{Ker} v \to \operatorname{Im} v$ *the associated algebraic isomorphism of* u *and* v, *respectively. We canonically identify* E *with* $(\ddot{E})'$ *and* G *with* $(\ddot{G})'$ *and define*

$$w_1 : (\ddot{G}/\operatorname{Ker} v)' \longrightarrow (\operatorname{Ker} v)^\circ, \quad a' \longmapsto r'a'$$

and let $w_2 : E/(\operatorname{Im} v)^\circ \to (\operatorname{Im} v)'$ *be the factorization of* j' *through* $E/(\operatorname{Im} v)^\circ$.

a) v, r, j, *and* $\overline{v}$ *are involutive.*

b) $\operatorname{Ker} u = (\operatorname{Im} v)^\circ$, $\operatorname{Im} u = (\operatorname{Ker} v)^\circ$, $\operatorname{Ker} v = {}^\circ(\operatorname{Im} u)$, $\operatorname{Im} v = {}^\circ(\operatorname{Ker} u)$.

c) w_1 *and* w_2 *are involutive isometries.*

d) $E/\operatorname{Ker} u$ *and* $\operatorname{Im} u$ *are* W^**-algebras,* q *aund* i *are* W^**-homomorphisms,* $\overline{u}$ *is an isometry of* W^**-algebras,* $\overline{v}$ *is an involutive isometry, and*

$$\overline{u} = w_1 \circ \overline{v}' \circ w_2 .$$

a) By Proposition 4.4.4.8 c), v is involutive. Hence $\operatorname{Ker} v$ is an involutive set of $\ddot{G}$ and $\operatorname{Im} v$ is an involutive set of F. By Proposition 2.3.2.22 o), $\overline{v}$ is involutive; j and r are obviously involutive.

b) By Theorem 4.2.6.6, $\operatorname{Im} u$ is closed and so by Proposition 1.4.2.11, $\operatorname{Im} v$ is closed as well. The assertion now follows from Corollary 1.4.2.10 a).

c) follows from a) and Proposition 2.3.2.22 g).

d) By b), $\operatorname{Ker} u$ and $\operatorname{Im} u$ are closed in E_F and $G_{\ddot{G}}$, respectively. By Corollary 4.4.4.12 d), $E/\operatorname{Ker} u$ is a W^*-algebra and q is a W^*-homomorphism. By Proposition 4.4.4.8 b) and Corollary 4.4.4.9 a $\Rightarrow$ b, $\operatorname{Im} u$ is a W^*-algebra and i is a W^*-homomorphism. By Theorem 4.2.6.6, $\overline{u}$ is an isometry of C^*-algebras. If by c), we identify $(\ddot{G}/\operatorname{Ker} v)'$ with $(\operatorname{Ker} v)^\circ = \operatorname{Im} u$ via w_1 and $E/(\operatorname{Im} v)^\circ = E/\operatorname{Ker} u$ with $(\operatorname{Im} v)'$ via w_2, then $\operatorname{Im} v$ becomes the predual of $E/\operatorname{Ker} u$, $\ddot{G}/\operatorname{Ker} v$ becomes the predual of $\operatorname{Im} u$, and $\overline{v}$ becomes the pretranspose of $\overline{u}$. By the above considerations, $\overline{u}$ is an isometry of W^*-algebras.

By Corollary 1.4.2.10 d),

$$\overline{u} = w_1 \circ \overline{v}' \circ w_2 ,$$

so that

$$\overline{v}' = w_1^{-1} \circ \overline{u} \circ w_2^{-1} .$$

By c), $\overline{v}'$ is an involutive isometry. Hence $\overline{v}$ is an involutive isometry as well (a) and Corollary 1.4.2.5). ∎

Proposition 4.4.4.17 *Let E, G be W^*-algebras and $u : E \to G$ a unital W^*-homomorphism. Take $x \in \operatorname{Sn} E$ and let μ be the spectral measure of x and f a bounded Borel function on $\sigma(x)$.*

a) $uf(x) = f(ux)$.

b) $u \circ \mu$ *is the spectral measure of* ux.

By Proposition 4.4.4.8 d), $u \circ \mu$ is a spectral measure. If $(A_\iota)_{\iota \in I}$ is a finite family of Borel sets of $\sigma(x)$ and $(\alpha_\iota)_{\iota \in I}$ is a family in $\mathbb{K}$, then

$$u\left(\sum_{\iota \in I} \alpha_\iota e_{A_\iota}(x)\right) = u\left(\sum_{\iota \in I} \alpha_\iota \mu(A_\iota)\right) = \sum_{\iota \in I} \alpha_\iota u \circ \mu(A_\iota) .$$

By continuity,

$$uf(x) = u\left(\int f d\mu\right) = \int f d(u \circ \mu)$$

(Theorem 4.3.2.19 b)). Hence $u \circ \mu$ is the spectral measure of ux and

$$uf(x) = f(ux) .$$ ∎

Theorem 4.4.4.18 (0) *We identify E with F' via the map*

$$E \longrightarrow F', \quad x \longmapsto \langle x, \cdot \rangle | F .$$

Let G be a C^*-subalgebra of E such that every continuous linear form on G is continuous on G_F. Put $H := (^\circ G)^\circ$ and let $j : G \to H$ be the inclusion map. Put $I := F/^\circ G$ and let $q : F \to I$ be the quotient map. Let u be the involutive isometry

$$I' \longrightarrow H, \quad x' \longmapsto q'x'$$

(Proposition 2.3.2.22 h)). If $G^\#$ is dense in $H_F^\#$, then:

a) *H is the closure of G in E_F and the W^*-subalgebra of E generated by G.*

b) *The map*

$$v : I \longrightarrow G', \quad a \longmapsto (j_I a) \circ u^{-1} \circ j$$

is an involutive isometry and

$$u \circ v' \circ j_G = j .$$

c) *$vqa = (j_F a)|G$ for every $a \in F$.*

d) *$(a, x) \in F \times G \Rightarrow x(vqa) = vq(xa)$, $(vqa)x = vq(ax)$.*

e) *$(a, x'') \in F \times G'' \Rightarrow \langle vqa, x'' \rangle = \langle a, uv'x'' \rangle$.*

f) *$(a, x'') \in F \times G'' \Rightarrow x''(vqa) = vq((uv'x'')a), (vqa)x'' = vq(a(uv'x''))$.*

g) *Given $x'', y'' \in G'' \Rightarrow$* $\begin{cases} uv'(x'' \vdash y'') = uv'(x'' \dashv y'') = (uv'x'')(uv'y''), \\ x'' \vdash y'' = x'' \dashv y''. \end{cases}$

h) *The Arens multiplications of G'' (Definition 2.2.7.13) coincide, G'' endowed with this multiplication is a W^*-algebra and $u \circ v' : G'' \to H$ is a W^*-isometry. If we identify G'' with H via $u \circ v'$, then j is the evaluation. If G is commutative, then G'' and H are also commutative.*

a) By Proposition 1.3.5.4, H is the closure of G in E_F. By Corollary 4.4.4.12 a), H is the W^*-subalgebra of E generated by G.

b) Since j_I is involutive (Proposition 2.3.2.22 e)), v is also involutive and the assertion follows from Proposition 1.3.6.21 d).

c) By Proposition 1.3.6.21 a),

$$\langle x, vqa\rangle = \langle x, (j_I qa) \circ u^{-1} \circ j\rangle = \langle x, a\rangle = \langle x, (j_F a)|G\rangle$$

for every $x \in G$. Thus

$$vqa = (j_F a)|G\,.$$

d) By c), Proposition 2.2.7.5, and Corollary 4.4.2.10,

$$\langle y, x(vqa)\rangle = \langle yx, vqa\rangle = \langle yx, j_F a\rangle = \langle y, xj_F a\rangle = \langle y, j_F(xa)\rangle = \langle y, vq(xa)\rangle\,,$$

$$\langle y, (vqa)x\rangle = \langle xy, vqa\rangle = \langle xy, j_F a\rangle = \langle y, (j_F a)x\rangle = \langle y, j_F(ax)\rangle = \langle y, vq(ax)\rangle$$

for every $y \in G$ and thus

$$x(vqa) = vq(xa)\,, \quad (vqa)x = vq(ax)\,.$$

e) We have

$$\langle vqa, x''\rangle = \langle a, q'v'x''\rangle = \langle a, uv'x''\rangle\,.$$

f) By b), d), and e),

$$\langle x, x''(vqa)\rangle = \langle (vqa)x, x''\rangle = \langle vq(ax), x''\rangle = \langle ax, uv'x''\rangle =$$

$$= \langle (uv'x'')a, x\rangle = \langle (uv'x'')a, jx\rangle = \langle (uv'x'')a, uv'j_G x\rangle =$$

$$= \langle vq((uv'x'')a), j_G x\rangle = \langle x, vq((uv'x'')a)\rangle\,,$$

$$\langle x, (vqa)x''\rangle = \langle x(vqa), x''\rangle = \langle vq(xa), x''\rangle = \langle xa, uv'x''\rangle =$$

$$= \langle a(uv'x''), x\rangle = \langle a(uv'x''), jx\rangle = \langle a(uv'x''\rangle, uv'j_G x\rangle =$$

$$= \langle vq(a(uv'x'')), j_G x\rangle = \langle x, vq(a(uv'x''))\rangle$$

for every $x \in G$. Thus

$$x''(vqa) = vq((uv'x'')a)\,, \quad (vqa)x'' = vq(a(uv'x''))\,.$$

g) By e) and f),

$$\langle a, uv'(x'' \vdash y'')\rangle = \langle vqa, x'' \vdash y''\rangle = \langle y''(vqa), x''\rangle = \langle vq((uv'y'')a), x''\rangle =$$

$$= \langle (uv'y'')a, uv'x''\rangle = \langle a, (uv'x'')(uv'y'')\rangle\,,$$

$$\langle a, uv'(x'' \dashv y'')\rangle = \langle vqa, x'' \dashv y''\rangle = \langle (vqa)x'', y''\rangle = \langle vq(a(uv'x'')), y''\rangle =$$

$$= \langle a(uv'x''), uv'y''\rangle = \langle a, (uv'x'')(uv'y'')\rangle$$

for every $a \in F$. Hence

$$uv'(x'' \dashv y'') = (uv'x'')(uv'y'') = uv'(x'' \vdash y'')\,,$$

$$x'' \dashv y'' = x'' \vdash y''\,.$$

h) By b) and Proposition 2.3.2.22 d), $u \circ v'$ is an involutive isometry. By g), the Arens multiplications of G'' coincide, G'' endowed with this multiplication is a C^*-algebra and $u \circ v' : G'' \to H$ is an isometry of C^*-algebras. By Theorem 2.2.7.15 e), G' is a G''-module and by Proposition 2.2.7.5, $\operatorname{Im} j_{G'}$ is a G''-submodule of G'''. Hence G'' is a W^*-algebra and it is easy to see that $u \circ v'$ is a W^*-isometry. By b), if we identify G'' and H via $u \circ v'$, then j becomes the evaluation. The last assertion follows from Theorem 2.2.7.15 g). ■

Proposition 4.4.4.19 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in F_+ such that $(\langle a_n, p\rangle)_{n\in\mathbb{N}}$ converges whenever $p \in \operatorname{Pr} E$. Then there is an $a \in F_+$ such that*

$$\lim_{n\to\infty} \langle a_n, x\rangle = \langle a, x\rangle$$

whenever $x \in E$.

By Corollary 4.4.1.14, there is an order continuous $a \in E'$ such that

$$\lim_{n\to\infty} \langle a_n, x\rangle = \langle a, x\rangle$$

whenever $x \in E$. Then $a \in E'_+$ (Corollary 4.4.1.7) and by Theorem 4.4.4.2 b $\Rightarrow$ a, $a \in F_+$. ■

Proposition 4.4.4.20 *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in F such that for each $p \in \operatorname{Pr} E$ $(\langle a_n, p\rangle)_{n\in\mathbb{N}}$ converges. If E is a Gelfand W^*-algebra, then there is an $a \in F$ such that*

$$\lim_{n\to\infty} \langle a_n, x\rangle = \langle a, x\rangle$$

whenever $x \in E$.

By Corollary 4.4.1.14, there is an order continuous $a \in E'$ such that

$$\lim_{n\to\infty} \langle a_n, x\rangle = \langle a, x\rangle$$

whenever $x \in E$. By Corollary 4.4.4.12 g), $a \in F$. ■

Remark. We do not know if the above proposition holds for W^*-algebras which are not Gelfand W^*-algebras.

Proposition 4.4.4.21 *Let $(E_t)_{t\in T}$ be a family of W^*-algebras and E the C^*-direct product of this family (Example 4.1.1.6). Put*

$$H := \left\{ a \in \prod_{t\in T} \ddot{E}_t \;\middle|\; \sum_{t\in T} \|a_t\| < \infty \right\}$$

and for $a \in H$ define

$$\widetilde{a} : E \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{t\in T} \langle x_t, a_t\rangle .$$

a) *E is a W^*-algebra and $\ddot{E} = \{\widetilde{a} \mid a \in H\}$.*

b) *For every $s \in T$, the maps*

$$p_s : E \longrightarrow E_s, \quad x \longmapsto x_s$$

$$q_s : E_s \longrightarrow E, \quad x_s \longmapsto (\delta_{st} x_s)_{t\in T},$$

are W^*-homomorphisms and $q_s(E_s)$ is a W^*- subalgebra of E.

a) It is easy to see that

$$\widetilde{F} := \{\widetilde{a} | a \in H\}$$

is an involutive E-submodule of E'. By Proposition 1.2.2.13, the map

$$E \longrightarrow \widetilde{F}', \quad x \longmapsto \langle x, \cdot\rangle | \widetilde{F}$$

is an isometry. Hence E is a W^*-algebra and $\ddot{E} = \widetilde{F}$.

b) By a), the maps

$$E_{\ddot{E}} \longrightarrow (E_s)_{\ddot{E}_s}, \quad x \longmapsto p_s x$$

$$(E_s)_{\ddot{E}_s} \longrightarrow E_{\ddot{E}}, \quad x \longmapsto q_s x$$

are continuous. Hence the maps p_s and q_s are W^*-homomorphisms (Proposition 4.4.4.6). By Corollary 4.4.4.8 b), $q_s(E_s)$ is a W^*-subalgebra of E. ■

Theorem 4.4.4.22 (0) *Let T be a nonempty compact space T. Then the following are equivalent:*

a) $\mathcal{C}(T)$ *is a W^*-algebra.*

b) T *is a Stone space and $\mathcal{C}(T)_{\mathcal{C}(T)^\pi}$ is Hausdorff.*

c) T *is hyperstonian.*

If these conditions are fulfilled, then:

d) $\mathcal{C}(T)^\pi$ *is the predual of $\mathcal{C}(T)$.*

e) *If x is a bounded Borel function on T and*

$$\widetilde{x} : \mathcal{C}(T)' \longrightarrow \mathbb{K}, \quad \mu \longmapsto \int x d\mu,$$

then $\widetilde{x} \in \mathcal{C}(T)'' = (\mathcal{C}(T)^\pi)'''$ and $\{x \neq j'\widetilde{x}\}$ is meager, where j denotes the evaluation of $\mathcal{C}(T)^\pi$.

a $\Rightarrow$ c follows from Corollary 4.4.1.10.

b $\Leftrightarrow$ c follows from Example 1.7.2.14 d).

c $\Rightarrow$ a & d. Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{C}(T)^\#$. Take $\mu \in \mathcal{C}(T)^\pi_+$. Then $\mathfrak{F}$ converges in $L^\infty(\mu)^\#_{L^1(\mu)}$ to a function x_μ. By Proposition 1.7.2.13 a $\Rightarrow$ b (and Example 1.7.2.14 a)), we may assume $x_\mu \in \mathcal{C}(T)^\#$.

Let $\mu, \nu \in \mathcal{C}(T)^\pi_+$. Then $\mu + \nu \in \mathcal{C}(T)^\pi_+$ and

$$x_\mu = x_{\mu+\nu} = x_\nu$$

on $(\mathrm{Supp}\,\mu) \cap (\mathrm{Supp}\,\nu)$. Hence we may define

$$x : S := \bigcup_{\mu \in \mathcal{C}(T)^\pi_+} \mathrm{Supp}\,\mu \longrightarrow \mathbb{K}, \quad t \longmapsto x_\mu(t),$$

where $t \in \mathrm{Supp}\,\mu$ for some $\mu \in \mathcal{C}(T)^\mu_+$, and x is continuous. Since T is homeomorphic to the Stone–Čech compactification of S, we may consider $x \in \mathcal{C}(T)^\#$. Then $\mathfrak{F}$ converges to x in $\mathcal{C}(T)^\#_{\mathcal{C}(T)^\pi}$ and $\mathcal{C}(T)^\#_{\mathcal{C}(T)^\pi}$ is compact.

By Proposition 1.3.6.27, $\mathcal{C}(T)$ is a W^*-algebra with $\mathcal{C}(T)^\pi$ as predual.

e) By Proposition 1.7.2.13 a $\Rightarrow$ b, there is a $y \in \mathcal{C}(T)$, such that $\{x \neq y\}$ is meager. Assume that $\{x \neq j'\widetilde{x}\}$ is not meager. Then $y \neq j'\widetilde{x}$. By Example 1.7.2.14 d), there is a $\mu \in \mathcal{C}(T)^\pi$, such that

$$\langle \mu, y \rangle \neq \langle \mu, j'\widetilde{x} \rangle .$$

By Example 1.7.2.14 a),

$$\langle j\mu, \widetilde{x}\rangle = \int x d(j\mu) = \int y d(j\mu) =$$

$$= \langle j\mu, y\rangle = \langle \mu, y\rangle \neq \langle \mu, j'\widetilde{x}\rangle = \langle j\mu, \widetilde{x}\rangle .$$

This is a contradiction. ■

Proposition 4.4.4.23 (8) *Let* $(x,y) \in \overset{\circ}{E}$. *If there is a filter* $\mathfrak{F}$ *on* E *such that*

$$\lim_{z,\mathfrak{F}} \langle (z,0),(a,b)\rangle = \langle (x,y),(a,b)\rangle$$

for any $(a,b) \in \overset{\cdot\cdot}{\overset{\circ}{E}}_+$ *then* $y = 0$ *and*

$$\lim_{z,\mathfrak{F}} \langle z,a\rangle = \langle x,a\rangle$$

for any $a \in \operatorname{Re} \ddot{E}$.

For every $(a,b) \in \overset{\cdot\cdot}{\overset{\circ}{E}}_+$,

$$\langle x,a\rangle - \langle y,b\rangle + i(\langle x,b\rangle + \langle y,a\rangle) =$$

$$= \langle (x,y),(a,b)\rangle = \lim_{z,\mathfrak{F}} \langle (z,0),(a,b)\rangle = \lim_{z,\mathfrak{F}} \langle z,\omega\rangle + i \lim_{z,\mathfrak{F}} \langle z,b\rangle .$$

In particular, for every $a \in \ddot{E}_+$,

$$\langle x,a\rangle + i\langle y,a\rangle = \lim_{z,\mathfrak{F}} \langle z,a\rangle$$

i.e.

$$\langle y,a\rangle = 0 ,$$

$$\langle x,a\rangle = \lim_{z,\mathfrak{F}} \langle z,a\rangle$$

(Proposition 4.3.6.1 $b_1 \Rightarrow b_2$). It follows

$$\langle y,b\rangle = 0 .$$

By Theorem 4.4.3.9 (and Corollary 4.4.2.10), $y = 0$ and

$$\lim_{z,\mathfrak{F}} \langle z,a\rangle = \langle x,a\rangle$$

for every $a \in \ddot{E}$. ■

Proposition 4.4.4.24 *Let E be a W^*-algebra and*

$$^cE := \{a \in \ddot{E} \mid x \in E \Rightarrow xa = ax\}\,.$$

Take $a \in {}^cE$ and let $(x, |a|)$ be its polar representation.

a) $x \in E^c\,, \quad |a| \in {}^cE\,.$

b) $(x, |a||E^c)$ *is the polar representation of* $a|E^c\,.$

c) $\|a|E^c\| = \|a\|\,.$

a) Case 1 $\mathbb{K} = \mathbb{C}$

Let $u \in Un\,E$ and put

$$b := u^*|a|u \in \ddot{E}_+$$

(Proposition 2.3.6.4). Then

$$ua = au = x|a|u = xub\,,$$

$$a = u^*xub\,.$$

Since

$$(xu)(xu)^* = xuu^*x^* = xx^* \in \Pr E\,,$$

it follows

$$(u^*xu)^*(u^*xu) = u^*x^*xu = (xu)^*(xu) \in \Pr E$$

(Corollary 4.1.2.22 b $\Rightarrow$ a). Moreover,

$$u^*x^*xub = u^*x^*x|a|u = u^*|a|u = b$$

(Theorem 4.4.3.5 a)). Let $y \in E$ such that

$$yb = 0\,.$$

Then

$$0 = yu^*x^*xub = yu^*x^*au\,,$$

$$0 = yu^*x^*a\,,$$

so that

$$0 = yu^*x^*xx^* = yu^*x^*$$

(Theorem 4.4.3.5 b), Corollary 4.1.2.22 a $\Rightarrow$ c),

$$yu^*x^*xu = 0\,.$$

Thus u^*x^*xu is the carrier of b and (xu, b) is the polar representation of ua. It follows that (u^*xu, b) is the polar representation of a. By the uniqueness of the polar representation,

$$u^*xu = x\,, \quad u^*|a|u = b = |a|\,,$$

so that

$$xu = ux\,, \quad |a|u = u|a|\,.$$

This implies $x \in E^c$ and $|a| \in {}^cE$ (Corollary 4.1.3.7).

Case 2 $\mathbb{K} = \mathbb{R}$

By complexification, we reduce this case to Case 1. We have

$$(a, 0) \in \overset{\circ}{\overbrace{{}^cE}} = {}^c(\overset{\circ}{E})$$

and it is easy to see that $((x, 0), (|a|, 0))$ is the polar representation of $(a, 0)$ (Corollary 4.4.3.7). By Case 1,

$$(x, 0) \in (\overset{\circ}{E})^c = \overset{\circ}{\overbrace{E^c}}\,, \quad (|a|, 0) \in {}^c(\overset{\circ}{E}) = \overset{\circ}{\overbrace{{}^cE}}\,,$$

and so

$$x \in E^c\,, \quad |a| \in {}^cE\,.$$

b) E^c is a W^*-subalgebra of E (Corollary 4.1.4.2 b), Corollary 4.4.2.5). Denote by

$$\varphi : E^c \longrightarrow E$$

the inclusion map and by

$$\ddot{\varphi} : \ddot{E} \longrightarrow \overset{\cdot\cdot}{\overbrace{E^c}}$$

its pretranspose (Corollary 4.4.4.8 c)). Then

$$a|E^c = \ddot{\varphi}a \in \overset{\cdot\cdot}{\widehat{E^c}}, \quad |a||E^c = \ddot{\varphi}|a| \in \overset{\cdot\cdot}{\widehat{E^c}},$$

so that (by a)), $(x, |a||E^c)$ is the polar representation of $a|E^c$.

c) By b) (and Corollary 2.3.4.7),

$$\|a|E^c\| = \| \, |a||E^c\| = \langle 1, |a| \rangle = \| \, |a| \, \| = \|a\| \, . \qquad \blacksquare$$

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